Option pricing and hedging under non-affine autoregressive stochastic volatility models

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Abstract

New pricing and hedging strategies are proposed for a non-affine auto-regressive stochastic factor model with non-predictable drift which allows to account for leverage effects. We consider a volatility dependent exponential linear pricing kernel with stochastic risk aversion parameters and implement both pricing and hedging for these models estimated via the hierarchical-likelihood method. This technique proves to outperform standard GARCH and Heston-Nandi based strategies in terms of a variety of considered criteria in an empirical exercise using historical returns and options data.

Keywords: autoregressive stochastic volatility models, bivariate Esscher transform, option pricing, local risk minimization hedging, h-likelihood estimation.

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1 Introduction

Empirical findings show strong evidence against several assumptions of the Black-Scholes (1973) option valuation model. The pricing and hedging of index options have been thus extensively studied in the context of stochastic volatility (SV) models in both discrete and continuous time.

In discrete-time settings, there are two main approaches for modeling the well documented volatility smile/smirk. The first direction is represented by the family of Generalized Autoregressive Conditionally Heteroskedastic (GARCH) models introduced by Engle (1982) and Bollerslev (1986), which have become very popular due to their ability to capture several of the "stylized facts" observed in financial markets, such as volatility clustering, fat tails, leverage effects, etc. Duan (1995) proposed a GARCH option pricing model driven by Gaussian innovations and based on a stochastic discount factor (SDF) which contains only a market price of equity risk. Due to its non-affine structure, there are no closed-form solutions available for pricing European style options. An alternative approach was provided by Heston and Nandi (2000), who introduced an affine class of Gaussian GARCH models which admit a semi-closed form expression for the unconditional Laplace transform of the log-asset price process. In order to improve the empirical fit of the Gaussian GARCH model, several extensions have been proposed in the literature, by including skewed and heavy-tailed innovations (see for instance Christoffersen *et al.* (2006), Chorro *et al.* (2012) and Badescu *et al.* (2008)), realized volatility measures (see, for example, Stentoft (2008), Hansen *et al.* (2011), Corsi *et al.* (2013) and Christoffersen *et al.* (2014)), multi-factor volatility dynamics (see Christoffersen *et al.* (2008) and Majewski *et al.* (2015)), variance dependent pricing kernels (see Christoffersen *et al.* (2013), Bormetti *et al.* (2015)) or combinations of the above (see Babaoglu *et al.* (2014) and Badescu *et al.* (2015)).

The second class consists of the Autoregressive Stochastic Volatility (ARSV) models (see Taylor (1986), Harvey *et al.* (1994) and Taylor (2005)), which can be viewed as discretizations of continuous-time SV models as they allow for separate driving noises for the asset-returns and the volatility/factor processes. Even though they provide more flexible dynamics than their GARCH counterparts, the ARSV models have not been as popular in terms of pricing and hedging financial derivatives, mainly due to estimation related issues, and only recently they got some attention by the research community. For example, Darolles *et al.* (2006) introduced the Compound Autoregressive (CAR) process which, equipped with an exponential affine pricing kernel, has been used for derivative valuation by Bertholon *et al.* (2008). The CAR framework has been extended by Khrapov and Renault (2014) who proposed an affine bivariate model for asset returns variance which allows for leverage effect and volatility feedback. Discrete-time affine stochastic volatility models with conditional skewness have been considered in Feunou and Tédongap (2012). They propose an ARSV option pricing model based on conditional Inverse Gaussian returns and autoregressive Gamma latent factors (see also Gourieroux and Jasiak (2006)). They show that these option pricing models outperform existing affine GARCH and continuous time jump diffusion models. However, the model parameters cannot be estimated using information coming from returns and options, since there is no existing link between the physical and the risk-neutral worlds.

The key ingredient in deriving pricing expressions in all aforementioned studies is the affine structure of the underlying models. However, the (affine) constraints typically imposed in the conditional mean return and volatility dynamics are somewhat restrictive. For example, Khrapov and Renault (2014) suppose that both the conditional Laplace transform of the variance process and the bivariate Laplace transform of the asset returns and variance process have an exponential affine form. Although this assumption leads to explicit or semi-explicit pricing formulae, there are not many distribution candidates for the driving noise process which satisfy the required condition. For instance, their empirical analysis is based on just one model constructed using a conditional Gaussian distribution for the asset returns and an autoregressive Gamma process for the latent factor. Another drawback of the affine models is that the exponential pricing kernel associated to the pricing methodology is typically based on constant equity and variance risk preference parameters, which is not consistent with empirical findings.

In order to address some of these issues, this paper proposes a simple one-factor non-affine ARSV option pricing model which allows for stochastic prices of risk and leverage effect. Our construction is based on a conditional Gaussian distribution for the factor process which drives both the conditional mean and variance of the asset returns. As in the affine modeling literature, we use an exponential pricing kernel which, in our situation, contains stochastic equity and factor risk premiums. Moreover, we show that having both risk premiums constant at the same time is not consistent with our setting, and that the market price of factor risk has a linear dependence of the factor process. We show that, unlike in the affine setting of Khrapov and Renault (2014), the risk-neutral bivariate Laplace transform has an exponential quadratic form in the latent factor. Moreover, our change of measure preserves the conditional distribution of the factor process, but not that of the asset returns.

Since the unconditional risk-neutral Laplace transform of the asset returns cannot be computed explicitly, and consequently a (spectral) generalized method of moments (GMM) estimation cannot be obtained using standard methods, we propose a sequential estimation procedure based on the information on both asset returns and option prices. First, using historical returns, we estimate the model parameters following the hierarchical likelihood (h-likelihood) technique proposed by Lee and Nelder (1996) and applied in the context of ARSV models by Lim *et al.* (2011). This method consists of performing a likelihood estimation in which the unobserved factor values are treated as unknown parameters in the optimization problem. In the second stage, the pricing kernel parameters are estimated based on the cross section of options written on the same underlying. We conduct an extensive empirical analysis to test the pricing and hedging performance of our model for a large panel of European put and calls options. The benchmark used in our analysis is the Heston and Nandi (2000) affine GARCH model. The option prices are computed based on Monte-Carlo simulation of weighted payoffs under the real-world measure, where the weights are given by the Radon-Nikodym derivatives of the corresponding measure change. Following the approach developed by Föllmer and Sondermann (1986), the hedging ratios are computed using the local risk minimization with respect to this risk-neutral measure. To facilitate the empirical comparison, we implement a similar estimation procedure for the affine GARCH model. Our results show that the ARSV model consistently outperforms its GARCH counterpart for almost all classes of moneyness and maturity considered. The overall Implied Volatility Root Mean Squared Error (IVRMSE) is improved by 24% in-sample and by 12% out-of-sample (in a weekly basis pricing exercise). Moreover, we notice that the average improvement in IVRMSE for in-the-money calls is around 34% in-sample and 26% for the out-of-sample weekly basis exercise. The only case when the GARCH performs slightly better than the ARSV is for long-term out-of-money contracts. Finally, a similar finding is revealed by the hedging exercise in the sense that the ARSV outperforms both the GARCH and the Black-Scholes model for all maturities and moneyness groups. The overall improvement over GARCH, as measured by a Normalized Hedging Error (NHE) indicator, is of 28%.

The paper is organized as follows. In Section 2 we introduce the underlying discrete-time stochastic volatility model. The pricing kernel and risk-neutral derivations are presented in Section 3. The estimation methodology and the numerical results are illustrated in Section 4. Section 5 concludes the paper.

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2 A non-affine one-factor autoregressive SV model

Consider a discrete time economy with the set of trading dates $\mathcal{T} = \{t | t = 0, 1 \ldots, T\}$, where T represents the terminal time. We assume that the market consists of a bond and a stock with corresponding price processes denoted by $B := \{B_t\}_{t\in\mathcal{T}}$ and $S := \{S_t\}_{t\in\mathcal{T}}$, respectively. Let r be the instantaneous constant risk free rate of return and assume that bond price dynamics is given by:

$$\begin{aligned} B_t &= e^{rt}, \;\; t \in \mathcal{T}, \\ B_0 &= 1. \end{aligned}$$

The log-return process, defined by $y := \{y_t\}_{t\in\mathcal{T}} = \{\log S_t/S_{t-1}\}_{t\in\mathcal{T}}$, is governed by the following one-factor autoregressive stochastic volatility model:

$$y_t = \mu + \lambda f_t^2 + f_t \epsilon_t, \quad \epsilon_t \sim \mathbf{N}(0,1), \tag{2.1}$$

$$f_t = \gamma + \phi f_{t-1} + \omega_t, \quad \omega_t \sim \mathbf{N}(0,\sigma_\omega^2). \tag{2.2}$$

We use the following assumptions and notations:

(1) The return process $y := \{y_t\}_{t\in\mathcal{T}}$ represents the observable quantity, while $f := \{f_t\}_{t\in\mathcal{T}}$ is an

unobservable one-dimensional latent factor process.[6]

(2) The innovations $\epsilon := \{\epsilon_t\}_{t\in\mathcal{T}}$ and $\omega := \{\omega_t\}_{t\in\mathcal{T}}$ are two sequences of i.i.d. Gaussian random variables which are both serially independent and also independent of each other.

(3) The model parameter vector is denoted by $\boldsymbol{\theta} := (\mu, \lambda, \gamma, \phi, \sigma_\omega) \in \mathbb{R}^5$ and is assumed to satisfy standard stationarity conditions.

(4) Let (Ω, P) be the probability space under which the model (2.1)-(2.2) has been introduced; we refer to P as the physical (real-world) probability measure. We denote by $\mathcal{F}_t := \sigma(y_s, f_s; s \leq t)$ the sigma-algebra generated by both the return and the factor processes, and introduce the augmented filtration $\mathcal{G}_t := \sigma(y_s, f_s, f_t; s \leq t-1) = \mathcal{F}_{t-1} \bigcup \{f_t\}$.

We notice that y_t is $\mathcal{F}_t$-measurable, but it is not $\mathcal{G}_t$-measurable, while, f_t is measurable with respect to both filtrations. Furthermore, the noise processes ϵ_t is independent of $\mathcal{G}_t$ and w_t is independent of $\mathcal{F}_{t-1}$. From equations (2.1)-(2.2), we can conclude that the asset returns are conditionally Gaussian distributed given $\mathcal{G}_t$, that is,

$$y_t|_{\mathcal{G}_t} \stackrel{P}{\sim} \mathbf{N}\left(\mu + \lambda f_t^2, f_t^2\right), \tag{2.3}$$

while the latent factor f_t has an $\mathcal{F}_{t-1}$-conditional Gaussian distribution, namely,

$$f_t|_{\mathcal{F}_{t-1}} \stackrel{P}{\sim} \mathbf{N}\left(m_{t-1}, \sigma_\omega^2\right), \tag{2.4}$$

with $m_t = \gamma + \phi f_t$.

We start by evaluating the joint cumulant generating function of the asset return and the factor process, conditionally on the information set $\mathcal{F}_t$.

Proposition 2.1 *If the asset returns follow the dynamics in (2.1)-(2.2), then the $\mathcal{F}_{t-1}$-conditional bivariate cumulant generating function of y_t and f_t under P is given by:*

$$C^P_{(y_t,f_t)}(z_1, z_2|\mathcal{F}_{t-1}) := \log \mathrm{E}^P\left[\exp(z_1 y_t + z_2 f_t)|\mathcal{F}_{t-1}\right] = z_1\mu - \frac{1}{2}\log u(z_1) + \frac{v(z_2, m_{t-1})}{u(z_1)} - \frac{m_{t-1}^2}{2\sigma_\omega^2}. \tag{2.5}$$

Here, $u(\cdot)$ and $v(\cdot, m_t)$ are two real-valued functions given by:

$$u(z_1) = 1 - z_1(z_1 + 2\lambda)\sigma_\omega^2, \tag{2.6}$$

$$v(z_2, m_t) = \frac{\sigma_\omega^2}{2}\left(z_2 + \frac{m_t}{\sigma_\omega^2}\right)^2, \tag{2.7}$$

which satisfy the conditions $u(0) = 1$ and $v(0, m_t) = m_t^2/(2\sigma_\omega^2)$.

Note that $C^P_{(y_t,f_t)}(z_1, z_2|\mathcal{F}_{t-1})$ is well defined only for the values z_1 such that $z_1(z_1 + 2\lambda)\sigma_\omega^2 < 1$ and that, furthermore, it is not an affine function of the factor process. In fact, if we expand m_{t-1} in (2.5), we

[6] This setting can be easily extended to the multi-factor case. In order to keep the presentation simple, we restrict our attention to the one-factor setup.

observe that $C^P_{(y_t,f_t)}(z_1,z_2|\mathcal{F}_{t-1})$ has a quadratic dependence with respect to f_{t-1}. We now characterize the univariate conditional cumulant generating functions of y_t and f_t. Indeed, taking $z_1 = 0$ in (2.5), it is easy to verify that the $\mathcal{F}_{t-1}$-conditional cumulant generating function of the latent factor corresponds to that of a Gaussian random variable with mean m_{t-1} and variance σ^2_ω:

$$C^P_{f_t}(z_2|\mathcal{F}_{t-1}) = C^P_{(y_t,f_t)}(0, z_2|\mathcal{F}_{t-1}) = z_2 m_{t-1} + \frac{1}{2} z_2^2 \sigma^2_\omega. \tag{2.8}$$

Similarly, the cumulant generating function of y_t conditionally on $\mathcal{F}_{t-1}$ can be obtained from (2.5) as follows:

$$C^P_{y_t}(z_1|\mathcal{F}_{t-1}) = C^P_{(y_t,f_t)}(z_1, 0|\mathcal{F}_{t-1}) = z_1\mu - \frac{1}{2}\log u(z_1) + \frac{m^2_{t-1}}{2\sigma^2_\omega}\left(\frac{1}{u(z_1)} - 1\right). \tag{2.9}$$

Taking the first and the second order derivatives of $C^P_{y_t}(z_1|\mathcal{F}_{t-1})$ evaluated at $z_1 = 0$, we obtain the conditional mean and variance of the asset returns under the physical measure:

$$\begin{aligned} \mathrm{E}^P[y_t|\mathcal{F}_{t-1}] &= \mu + \lambda\left(m^2_{t-1} + \sigma^2_\omega\right), && (2.10)\\ \mathrm{Var}^P[y_t|\mathcal{F}_{t-1}] &= m^2_{t-1} + \sigma^2_\omega + 2\lambda^2\sigma^2_\omega\left(2m^2_{t-1} + \sigma^2_\omega\right). && (2.11) \end{aligned}$$

The conditional covariance between the asset return process and the latent factor is given by:

$$\mathrm{Cov}^P(y_t, f_t|\mathcal{F}_{t-1}) = 2\lambda m_{t-1}\sigma^2_\omega. \tag{2.12}$$

Finally, we can compute the covariance between the asset return process and its one-step ahead conditional variance given the information set $\mathcal{F}_{t-1}$ as:

$$\mathrm{Cov}^P\left(y_t, \mathrm{Var}^P[y_t|\mathcal{F}_{t-1}]\,|\mathcal{F}_{t-1}\right) = 2\lambda\phi\sigma^2_\omega(1 + 4\lambda^2\sigma^2_\omega)\left[2\gamma m_{t-1} + \phi\left(2m^2_{t-1} + \sigma^2_\omega\right)\right]. \tag{2.13}$$

Thus, both parameters λ and ϕ contribute to the leverage effect.

3 Pricing kernel and risk-neutral dynamics

The choice of the risk-neutral measure plays an important role in the pricing and hedging of financial derivatives. In this section we introduce an exponential linear pricing kernel via a bivariate conditional Esscher transform.

First, for any $t \in \mathcal{T}$, we define the following stochastic process $N = \{N_t\}_{t\in\mathcal{T}}$:

$$N_t = \exp\Big(\eta_{1t}y_t + \eta_{2t}f_t - C^P_{(y_t,f_t)}(\eta_{1t}, \eta_{2t}|\mathcal{F}_{t-1})\Big). \tag{3.1}$$

Here, $\eta_{1t} = \{\eta_{1t}\}_{t\in\mathcal{T}}$ and $\eta_{2t} = \{\eta_{2t}\}_{t\in\mathcal{T}}$ are two $\mathcal{F}_t$-predictable processes representing the market prices of equity risk and factor risk, respectively. Similar stochastic discount factors have been proposed in the context of discrete-time stochastic volatility models (see e.g. Khrapov and Renault (2014) and Bormetti *et al.* (2015) among others), but only with constant market prices of risk. However, as showed later in this section, it is not possible to have both η_{1t} and η_{2t} constants in our setting.

The pricing measure Q is introduced through the following Radon-Nikodym derivative process denoted by $Z = \{Z\}_{t\in\mathcal{T}}$:

$$\left.\frac{dQ}{dP}\right|_{\mathcal{F}_T} := Z_T = \prod_{t=1}^{T} N_t = \prod_{t=1}^{T} \exp\Big(\eta_{1t}y_t + \eta_{2t}f_t - C^P_{(y_t,f_t)}\left(\eta_{1t},\eta_{2t}|\mathcal{F}_{t-1}\right)\Big). \tag{3.2}$$

It is straightforward to check that Q is an equivalent probability measure with respect to P. Indeed, we first notice that Z_t defined in (3.2) with $Z_0 = 1$ is an $\mathcal{F}_t$-martingale:

$$\mathrm{E}^P\left[Z_t|\mathcal{F}_{t-1}\right] = Z_{t-1}\mathrm{E}^P\left[N_t|\mathcal{F}_{t-1}\right] = Z_{t-1}\mathrm{E}^P\Big[\exp\Big(\eta_{1t}y_t + \eta_{2t}f_t - C^P_{(y_t,f_t)}\left(\eta_{1t},\eta_{2t}|\mathcal{F}_{t-1}\right)\Big)|\mathcal{F}_{t-1}\Big] = Z_{t-1}.$$

Since Z_T is positive by definition and $\mathrm{E}^Q[1] = \mathrm{E}^P\left[Z_T\right] = \mathrm{E}^P\left[Z_0\right] = 1$, it follows that Q is a well-defined probability measure equivalent to P. In order for Q to be a risk-neutral measure, we need to impose a constraint on the market prices of risk such that the discounted asset prices are martingales after the change of measure. To achieve this, we characterize the conditional bivariate cumulant generating function of the asset returns and latent factor under the pricing measure defined above.

Proposition 3.1 *If the asset returns follow the dynamics in (2.1)-(2.2), then the $\mathcal{F}_{t-1}$-conditional bivariate cumulant generating function of y_t and f_t under Q defined in (3.2) is given by:*

$$C^Q_{(y_t,f_t)}\left(z_1,z_2|\mathcal{F}_{t-1}\right) = z_1\mu - \frac{1}{2}\log\frac{u\left(z_1+\eta_{1t}\right)}{u\left(\eta_{1t}\right)} + \frac{v\left(z_2+\eta_{2t},m_{t-1}\right)}{u\left(z_1+\eta_{1t}\right)} - \frac{v\left(\eta_{2t},m_{t-1}\right)}{u\left(\eta_{1t}\right)}. \tag{3.3}$$

Here, the functions $u(\cdot)$ and $v(\cdot,m_t)$ are those defined in (2.6)-(2.7).

Using the above representation, we can re-write the martingale constraint for the discounted asset price, $\mathrm{E}^Q\left[\exp y_t|\mathcal{F}_{t-1}\right] = \exp r$, as, $C^Q_{(y_t,f_t)}\left(1,0|\mathcal{F}_{t-1}\right) = \exp r$. Thus, it follows that for any $t\in\mathcal{T}$, the market prices of risk η_{1t} and η_{2t} must satisfy the equation below:

$$\mu - r - \frac{1}{2}h_1\left(1,\eta_{1t}\right) + v\left(\eta_{2t},m_{t-1}\right)h_2\left(1,\eta_{1t}\right) = 0, \tag{3.4}$$

where $h_1\left(\cdot,\eta_{1t}\right)$ and $h_2\left(\cdot,\eta_{1t}\right)$ are the two real-valued functions given by:

$$h_1\left(z_1,\eta_{1t}\right) := \log\frac{u\left(z_1+\eta_{1t}\right)}{u\left(\eta_{1t}\right)}, \tag{3.5}$$

$$h_2\left(z_1,\eta_{1t}\right) := \frac{1}{u\left(z_1+\eta_{1t}\right)} - \frac{1}{u\left(\eta_{1t}\right)}. \tag{3.6}$$

Unlike in the univariate conditional Esscher transform case where the market price of risk is uniquely determined by the martingale constraint, here we have an infinite number of pairs (η_{1t},η_{2t}) which solve

(3.4). In general, we need to use option prices in order to calibrate for one of these parameters. For example, we notice that using (3.4) we can solve for η_{2t} as a function of η_{1t} as follows:

$$\eta_{2t} = -\frac{m_{t-1}}{\sigma_\omega^2} \pm \sqrt{\frac{2}{\sigma_\omega h_2\left(1, \eta_{1t}\right)} \left(r - \mu + \frac{1}{2} h_1\left(1, \eta_{1t}\right)\right)}. \tag{3.7}$$

If we let η_{1t} to be a constant for any $t \in \mathcal{T}$, this results in having an η_{2t} which depends on f_{t-1}. Therefore, we cannot have both market prices of risk to be constant at the same time. In our empirical applications we shall assume a constant factor preference parameter which will be estimated from observed option quotes, while we let the market price of equity risk to be a stochastic process determined by (3.4). In order to potentially identify the risk-neutral distributions of the asset return process and the latent factor conditional on $\mathcal{F}_t$, we need to evaluate the corresponding cumulant generating functions. This is carried out in the following corollaries.

Corollary 3.1 *If the asset returns follow the dynamics in (2.1)-(2.2), then the $\mathcal{F}_{t-1}$-conditional cumulant generating function of the latent factor f_t under Q defined in (3.2) is given by:*

$$C_{f_t}^{Q}\left(z_2 | \mathcal{F}_{t-1}\right) = z_2 \frac{\sigma_\omega^2}{u(\eta_{1t})} \left(\eta_{2t} + \frac{m_{t-1}}{\sigma_\omega^2}\right) + \frac{z_2^2}{2} \frac{\sigma_\omega^2}{u(\eta_{1t})}. \tag{3.8}$$

The proof follows immediately by setting $z_1 = 0$ in (3.3).

We notice that the expression (3.8) corresponds to the cumulant generating function of a Gaussian random variable and hence we can conclude that:

$$f_t\Big|_{\mathcal{F}_{t-1}} \overset{Q}{\sim} \mathbf{N}\left(\frac{\sigma_\omega^2}{u(\eta_{1t})}\left(\eta_{2t} + \frac{m_{t-1}}{\sigma_\omega^2}\right), \frac{\sigma_\omega^2}{u(\eta_{1t})}\right).$$

Thus, the risk-neutral measure Q preserves the underlying conditional distribution of the latent factor. Moreover, if we set η_{1t} to be constant, $\eta_{1t} = \eta_1$, it follows from (3.7) that f_t has an $AR(1)$ structure under Q.

Corollary 3.2 *If the asset returns follow the dynamics from (2.1)-(2.2), then the $\mathcal{F}_{t-1}$-conditional cumulant generating function of y_t under Q defined in (3.2) is given by:*

$$C_{y_t}^{Q}\left(z_1 | \mathcal{F}_{t-1}\right) = z_1 \mu - \frac{1}{2} h_1\left(z_1, \eta_{1t}\right) + \frac{h_2\left(z_1, \eta_{1t}\right)}{h_2\left(1, \eta_{1t}\right)} \left(r - \mu + \frac{1}{2} h_1\left(1, \eta_{1t}\right)\right), \tag{3.9}$$

where $h_1\left(z_1, \eta_{1t}\right)$ and $h_2\left(z_1, \eta_{1t}\right)$ are given in (3.5)-(3.6).

The proof follows immediately by replacing $z_2 = 0$ into (3.3) and from using the martingale constraint (3.4).

In this case, we notice that the pricing measure does not preserve the underlying distribution of the asset returns since the above risk-neutral cumulant generating function is not of the same form as that in (2.9). However, in order to perform pricing and hedging of financial derivatives one does not necessarily need to use the asset return dynamics under the martingale measure. In the next section we show how options can be computed under the physical measure by making use of the closed-form expression of the Radon-Nikodym derivative in (3.2).

4 Empirical analysis

In this section we investigate the pricing and hedging performance of the autoregressive SV in (2.1)-(2.2) when risk-neutralized via the exponential factor dependent pricing kernel introduced in Section 3. The model based option prices are computed based on a sequential estimation procedure in which the model parameters are obtained using a h-likelihood technique that uses historical returns, while the pricing kernel parameter is estimated by maximizing an option likelihood function in which quoted option prices intervene. The hedging ratios are constructed using local-risk minimization strategies under the risk-neutral measure. The pricing and hedging performance of our model is conducted using an extensive dataset of European calls and puts on the S&P 500 index and it is tested relative to the Heston-Nandi GARCH pricing model that is used as a benchmark.

4.1 Data description

We investigate the pricing performance using three datasets of S&P500 European options. The first two contain call option quotes ranging over the period January 1st, 2012–December 31st, 2013. Both datasets comprise contracts with maturities between 20 and 250 days and moneyness between 0.9 and 1.1 and were obtained after applying standard filters similar to those proposed in Bakshi *et al.* (1997). The first dataset, called Sample A, contains of 6,047 call prices quoted every Wednesday of the reference period considered and is used for the in and out-of-sample analysis, while the second dataset, called Sample B, consists of 6,315 call prices recorded every Thursday for the same period and is only used for the out-of-sample performance assessment. The basic features of the datasets which include the number of contracts, average prices, and implied volatilities are illustrated in Tables 1 and 2. The average price and implied volatility for Sample A are \$26.683 and 14.0%, respectively, while the corresponding values for Sample B are \$27.33 and 14.0%, respectively. All options are grouped into six classes of moneyness and four classes of maturities.[7] Finally, for the hedging exercise we use a third dataset, called Sample C, which consists of 2,728 S&P500 put options whose main features are described in Table 3.

4.2 Estimation methodology

The model parameters are estimated using a two-stage procedure based on both asset returns and the option data described in the previous subsection.

[7] The moneyness is defined as the ratio between the price of the underlying S_0 and the strike price K ($Mo := F/K$), so call (respectively, put) options with $Mo < 1$ (respectively, $Mo > 1$) are out-of-money (OTM) and those with $Mo > 1$ (respectively, $Mo < 1$) are in-the-money (ITM).

4.2.1 The h-likelihood estimation using returns

Historical return data on S&P 500 is used to estimate the parameter vector $\boldsymbol{\theta} = (\mu, \lambda, \gamma, \phi, \sigma_\omega)$ of the underlying autoregressive SV model in (2.1)-(2.2). We start by noticing that the model (2.1)-(2.2) is a random-effect model for which the hierarchical GLM framework, first introduced in Lee and Nelder (1996) and Lee and Nelder (2006), provides an easy-to-implement h-likelihood based estimation technique. This method with its various extensions (see for instance del Castillo and Lee (2008); Lim et al. (2011) for its implementation in the context of ARSV models) consists in writing the hierarchical likelihood as a function of the unknown parameters and of the random effects, which in our particular case are represented by the latent factor values, and in its subsequent maximization. Estimates for both the model parameters and the latent variables are hence obtained as the solutions of the corresponding optimization problem. In the following paragraphs we describe in detail the adopted h-likelihood estimation method implemented for the model specification in (2.1)-(2.2).

Let $\mathbf{y} := (y_1, y_2, \ldots, y_T)^\top$ and $\mathbf{f} := (f_1, f_2, \ldots, f_T)^\top$ be vectors containing T observed returns and T corresponding unobserved latent factor values, respectively. Denote by $g(\mathbf{y}, \mathbf{f})$ their joint probability density function and define the associated h-loglikelihood function as in Lee and Nelder (1996) and Lee and Nelder (2006), namely,

$$h(\mathbf{y}; \mathbf{f}, \boldsymbol{\theta}) := \log g(\mathbf{y}, \mathbf{f}) = \sum_{t=1}^{T} \log g(y_t, f_t | \mathcal{F}_{t-1}), \tag{4.1}$$

where we recall that $\mathcal{F}_t = \sigma(y_s, f_s; s \leq t)$ is the sigma-algebra generated by both the returns and the factor values up to time t. Additionally, we notice that for any $t \in \{1, \ldots, T\}$ it holds that

$$\log g(y_t, f_t | \mathcal{F}_{t-1}) = \log g(y_t | \mathcal{G}_t) + \log g(f_t | \mathcal{F}_{t-1}), \tag{4.2}$$

where we used the fact that the innovations $\{\epsilon_t\}_{t \in \mathcal{T}}$ and $\{\omega_t\}_{t \in \mathcal{T}}$ are assumed to be independent (see Assumption (2)). The two summands in (4.2) are easy to obtain as the logarithms of the probability density functions for the conditional Gaussian distributions in (2.3) and in (2.4), respectively. Using (4.2) with (2.3) and (2.4) in (4.1), we can write the h-loglikelihood function as

$$h(\mathbf{y}; \mathbf{f}, \boldsymbol{\theta}) = -T \log(2\pi) - \frac{1}{2} \left\{ T \log \sigma_\omega^2 + \sum_{t=1}^{T} \frac{\left(y_t - \mu - \lambda f_t^2\right)^2}{f_t^2} + \frac{1}{\sigma_\omega^2} \sum_{t=1}^{T} (f_t - \gamma - \phi f_{t-1})^2 + \sum_{t=1}^{T} \log(f_t^2) \right\}, \tag{4.3}$$

where we assume the existence of an initial value $f_0 \in \mathbb{R}$. In our empirical estimation experiment, as it is common in econometrics (see Pollock (2003)), the model is initialized by setting f_0 equal to the unconditional first moment of the latent stationary stochastic process f that is, $f_0 := \mathrm{E}[f] = \dfrac{\gamma}{1-\phi}$.

Following further the h-likelihood technique, we define the vector of smoothed latent factor values as $\mathbf{f}_{\cdot|T} := \left(f_{1|T}, f_{2|T}, \ldots, f_{T|T}\right)$ with $f_{t|T} := \mathrm{E}[f_t | \mathcal{F}_T]$, $t \in \{1, \ldots, T\}$. The fact that the distribution for the innovations is assumed to be Gaussian and, in particular, symmetric and unimodal, allows us to obtain the components of $\mathbf{f}_{\cdot|T}$ as the extrema of the corresponding density functions or, equivalently, of the h-loglikelihood function in (4.3). More specifically, estimates $\widehat{\mathbf{f}_{\cdot|T}}$ are provided by solution of the optimization problem

$$\widehat{\mathbf{f}_{\cdot|T}} = \underset{\mathbf{f}\in\mathbb{R}^T}{\arg\max}\ h(\mathbf{y};\mathbf{f},\boldsymbol{\theta}). \tag{4.4}$$

It is of a common practice to tackle this problem solving the score equation $\nabla_{\mathbf{f}} h(\mathbf{y};\mathbf{f},\boldsymbol{\theta}) = \mathbf{0}_T$, with $\nabla_{\mathbf{f}} h(\mathbf{y};\mathbf{f},\boldsymbol{\theta}) := \left(\frac{\partial h(\mathbf{y};\mathbf{f},\boldsymbol{\theta})}{\partial f_1}, \frac{\partial h(\mathbf{y};\mathbf{f},\boldsymbol{\theta})}{\partial f_2}, \ldots, \frac{\partial h(\mathbf{y};\mathbf{f},\boldsymbol{\theta})}{\partial f_T}\right)^{\top}$, using the Fischer scoring method (we refer the reader to Lim et al. (2011) and the appendix therein for sparsity-based techniques that can be used to speed up this computation). In order to make our discussion self-contained, we provide in the appendix explicit expressions of the gradient and the entries of the associated Hessian matrix.

Finally, with the estimates $\widehat{\mathbf{f}_{\cdot|T}}$ of the smoothed latent factor values $\mathbf{f}_{\cdot|T}$ determined in (4.4), the fixed model parameters $\boldsymbol{\theta}$ are estimated using the corresponding adjusted profile h-loglikelihood function introduced in Lee and Nelder (1996) as

$$h_{\mathrm{profile}}(\boldsymbol{\theta}) := h\left(\mathbf{y};\widehat{\mathbf{f}_{\cdot|T}}(\boldsymbol{\theta}),\boldsymbol{\theta}\right) - \frac{1}{2}\log\det\frac{H\left(\mathbf{y};\widehat{\mathbf{f}_{\cdot|T}}(\boldsymbol{\theta}),\boldsymbol{\theta}\right)}{2\pi}, \tag{4.5}$$

where $H\left(\mathbf{y};\widehat{\mathbf{f}_{\cdot|T}}(\boldsymbol{\theta}),\boldsymbol{\theta}\right)$ is the opposite of the Hessian matrix of the h-loglikelihood evaluated at the indicated values and whose elements are provided in the appendices. The estimate $\widehat{\boldsymbol{\theta}}$ is computed as the solution of the optimization problem

$$\widehat{\boldsymbol{\theta}} = \underset{\boldsymbol{\theta}\in\mathbb{R}^5}{\arg\max}\ h_{\mathrm{profile}}(\boldsymbol{\theta}). \tag{4.6}$$

As a summary, the h-likelihood estimation steps are:

(1) Given a log-returns sample $\mathbf{y}$ and a current estimate $\widehat{\boldsymbol{\theta}}$ of $\boldsymbol{\theta}$, the estimates $\widehat{\mathbf{f}_{\cdot|T}}$ of the smoothed latent factor values $\mathbf{f}_{\cdot|T}$ are computed by solving (4.4);

(2) Given the estimates $\widehat{\mathbf{f}_{\cdot|T}}$, the estimates $\widehat{\boldsymbol{\theta}}$ of the parameter vector $\boldsymbol{\theta}$ are updated by maximizing the adjusted profile h-likelihood as in (4.6).

Again, we refer the reader to the appendices for all the necessary details regarding gradient and Hessian computations as well as the solution of (4.4). The appendix also contains information on how to use the h-likelihood framework in order to filter and forecast the factor values.

The estimates $\widehat{\mathbf{f}_{\cdot|T}}$ and $\widehat{\boldsymbol{\theta}}$ of the smoothed latent factor values $\mathbf{f}_{\cdot|T}$ and of the parameter vector $\boldsymbol{\theta}$, respectively, allow us to proceed to a second stage in which we calibrate the pricing kernel parameters using quoted option prices.

4.2.2 Pricing kernel calibration using option prices

In the second stage, given the parameter values estimated in the h-likelihood procedure, we calibrate the pricing kernel parameters using the observed option prices. More specifically, we assume a constant factor preference parameter, $\eta_{2t} = \eta_2$, and since the market of price of factor risk η_{1t} depends on η_2 through equation (3.4), we have only one parameter to be estimated at this step.

Following the approach in Trolle and Schwartz (2009), we construct an option likelihood in the following way. For each set of quoted option prices $\mathbf{O}^{\text{mkt}} := \{O_1^{\text{mkt}}, \ldots, O_N^{\text{mkt}}\}$, we define the vega weighted option errors by:

$$e_i := \frac{O_i^{\text{mkt}} - O_i}{\nu_i^{\text{mkt}}}, \quad i = 1, \ldots, N.$$

Here, O_i represents the model option price and ν_i^{mkt} is the Black-Scholes vega corresponding to the quoted option price. Furthermore, we assume that e_i are independent and normally distributed with mean zero and variance $\sigma_e^2 := \frac{1}{N} \sum_{i=1}^{N} e_i^2$. Thus, the log-likelihood function of the option price vector $\mathbf{O}^{\text{mkt}}$ is given by:

$$\log L(\mathbf{O}^{\text{mkt}}, \widehat{\boldsymbol{\theta}}, \eta_2) = \sum_{i=1}^{N} \log L_i(O_i^{\text{mkt}}, \widehat{\boldsymbol{\theta}}, \eta_2) = \log\left(\frac{1}{\nu_i^{\text{mkt}}} f_e\left(\frac{O_i^{\text{mkt}} - O_i}{\nu_i^{\text{mkt}}}\right)\right),$$

where $f_e(\cdot)$ is the probability density function for a Gaussian random variable with mean zero and variance σ_e^2, $\widehat{\boldsymbol{\theta}}$ is the estimated model parameter vector from the h-likelihood step, and η_2 is the pricing kernel parameter to be estimated. In the remainder of this subsection, we briefly describe how the option prices are computed.

Note that unlike the situation under the physical measure, there are no explicit expressions available that describe the risk-neutral dynamics of the asset returns under the factor-dependent pricing kernel. Therefore, we evaluate the option prices using Monte-Carlo simulations under P by generating the asset paths according to (2.1)-(2.2) and then weighting the option payoff by the corresponding Radon-Nikodym derivative path. For example, the price of any call option i with $i = 1, \ldots, N$, strike K_i and maturity T_i is given by:

$$O_i = \mathrm{E}^Q\left[\exp\left(-rT_i\right) \max\left(S_{T_i} - K_i, 0\right) | \mathcal{F}_{t-1}\right] = \frac{1}{M} \sum_{j=1}^{M} \exp\left(-rT_i\right) \max\left(S_{T_i}^P(j) - K_i, 0\right) N_{T_i}(j).$$

Here $S_{T_i}^P(j),\ j = 1, \ldots, M$ represents the j-th simulated path of the asset price under the physical measure, $N_{T_i}(j)$ is the j-th simulated Radon-Nikodym factor given in (3.2), and M is the number of Monte-Carlo paths. In our calibration exercise we use $M = 20,000$. The continuously compounded one-period interest rate r is obtained from the corresponding T-Bill rates adequately interpolated in order to match the option maturity. The starting values for the asset price process and the latent value are provided by the last smoothing estimates obtained from the h-likelihood procedure. Furthermore, in order to reduce stochastic noise and in the spirit of Eichler *et al.* (2011), we use the same random numbers for generating the Monte Carlo paths at each step in the maximization of the likelihood. Note that, in general, $\log L(\mathbf{O}^{\mathrm{mkt}}, \widehat{\boldsymbol{\theta}}, \eta_2)$ is a non-convex function of η_2, so we need to use global search solvers in order to make sure that we end up with a global optimal solution.

4.3 Hedging implementation

We follow a hedging strategy based on the minimization of the local risk, as presented in Fölmer *et al.* (2002) and references therein. This hedging implementation is based on the construction of two-instrument portfolios formed with the risk asset and a risk-free bond. We consider a generalized trading strategy denoted by $(\xi^B, \xi^S) = \left\{\left(\xi_t^B, \xi_t^S\right)\right\}_{t\in\mathcal{T}}$, where ξ_t^B is adapted to the filtration $\mathcal{F}_t$ and represents the amount invested in the bond, while ξ^S is a $\mathcal{F}_t$-predictable process quantifying the amount invested in the risky asset. The value process associated to this trading strategy, $V(\xi^B, \xi^S) = \left\{V_t(\xi^B, \xi^S)\right\}_{t\in\mathcal{T}}$, is defined by:

$$V_0(\xi^B, \xi^S) = \xi_0^B, \quad \text{and} \quad V_t(\xi^B, \xi^S) = \xi_t^B \cdot B_t + \xi_t^S \cdot S_t, \quad t \in \mathcal{T}.$$

The cost process associated to this strategy is denoted by $C(\xi^B, \xi^S) = \left\{C_t(\xi^B, \xi^S)\right\}_{t\in\mathcal{T}}$ and is given by:

$$C_t(\xi^B, \xi^S) = V_t(\xi^B, \xi^S) - \sum_{k=1}^{t} \xi_k^S \cdot (S_k - S_{k-1}), \quad t \in \mathcal{T}.$$

Since under our autoregressive SV model the markets are incomplete, option prices cannot be fully replicated using such self-financing portfolios. There are various optimality criteria proposed in the literature to tackle simultaneously the pricing and hedging of these financial derivatives. In this paper we use the local risk minimization criteria, where the optimization problem is carried out in the risk neutral world.[8] More specifically, we find the trading strategy which solves the following optimization problem for any $t \in \mathcal{T}$:

$$(\widehat{\xi}_t^B, \widehat{\xi}_t^S) = \underset{\xi_t^B, \xi_t^S}{\operatorname{argmin}} \mathrm{E}^Q \left[\left(\widetilde{C}_{t+1}(\xi^B, \xi^S) - \widetilde{C}_t(\xi^B, \xi^S) \right)^2 \middle| \mathcal{F}_t \right], \quad t \in \mathcal{T}.$$

Here, $\widetilde{C}(\xi^B, \xi^S)_t$ represents the discounted cost of hedging $\widetilde{C}(\xi^B, \xi^S)_t := \exp(-rt) C(\xi^B, \xi^S)_t$. The optimal locally risk minimizing trading strategy for a financial derivative with payoff $H(S_T)$ at maturity is determined by the following recursions:

$$\widehat{\xi}_{t+1}^S = \exp(-r(T-t)) \frac{\mathrm{E}^Q \left[H(S_T) \left(\exp(-r) S_{t+1} - S_t \right) \mid \mathcal{F}_t \right]}{\mathrm{Var}^Q \left[\exp(-r) S_{t+1} - S_t \mid \mathcal{F}_t \right]}, \tag{4.7}$$

$$\widehat{V}_t(\widehat{\xi}^B, \widehat{\xi}^S) = \mathrm{E}^Q \left[\exp(-r(T-t)) H(S_T) \mid \mathcal{F}_t \right], \tag{4.8}$$

[8] A detailed discussion on the advantages of using the local risk minimization criteria with respect to a martingale measure is provided in Badescu *et al.* (2014).

Here we assumed that $\widehat{V}_T(\widehat{\xi}^B, \widehat{\xi}^S) = H(S_T)$ and for any $t \in \mathcal{T}$, the optimal allocation in the bond is determined from:

$$\widehat{\xi}_t^B = \frac{1}{B_t}\left(\widehat{V}_t(\widehat{\xi}^B, \widehat{\xi}^S) - \widehat{\xi}_t^S S_t\right), \tag{4.9}$$

where $\widehat{\xi}_t^S$ and $\widehat{V}_t(\widehat{\xi}^B, \widehat{\xi}^S)$ are provided in (4.7)-(4.8). Note that the above scheme requires that the hedging is performed on a daily basis. However, the above recursions can be easily adapted if hedging takes place at lower frequencies than the one at which the asset prices are observed. For example, in our empirical analysis, the optimal portfolio is rebalanced on a weekly basis even though the prices of the underlying asset are quoted daily.

Since there are no closed-form expressions available, the hedging ratios $\widehat{\xi}^S$ are evaluated using Monte-Carlo simulations. Thus, the conditional expectation and variance in (4.7) are estimated using $M = 5,000$ paths, at each time $t \in \mathcal{T}$. Each strategy is initialized using the estimated parameters and the smoothed volatilities coming from the h-likelihood step.

4.4 Empirical results

The empirical pricing and hedging performance of our autoregressive SV (ARSV) model is assessed in this section. For the pricing exercise we consider as benchmark the affine GARCH model of Heston and Nandi (2000) (HNGARCH) risk-neutralized with the variance dependent pricing kernel of Christoffersen *et al.* (2013)[9]. In the hedging analysis we additionally include in the comparison the Black-Scholes model performance.

4.4.1 Option pricing performance

We carry out an extensive in and out-of-sample pricing performance assessment using the option data sets spelled out in Tables 1 and 2. The model and the pricing kernel parameters are estimated using the sequential estimation procedure described in Section 4.2 and are updated as follows. For the first day in Sample A (Wednesdays), we run the h-likelihood estimation of the model parameter vector $\boldsymbol{\theta} = (\mu, \lambda, \gamma, \phi, \sigma_\omega)$ using the historical daily returns on S&P 500 over a period of ten years prior to that date, for a total of 2,520 observations. Next, we calibrate the latent factor risk parameter η_2 from the pricing kernel using an options likelihood constructed using all options quotes on that particular Wednesday. Finally, the parameters obtained at this stage are used for the out-of-sample exercise in which we compute the prices quoted the next day (the corresponding Thursday from Sample B), as well as those quoted the next Wednesday from Sample A. This procedure is repeated for the whole dataset from Sample A and the model parameters are reestimated on a monthly basis using a rolling window of 2,520 observations.

[9]Note that although the risk-neutral GARCH model of Heston and Nandi (2000) allows for a semi-closed form expression for the option prices, we implement it using Monte Carlo simulations in a similar fashion as our autoregressive SV model, in order to make the comparison more fair.

In order to assess the performance of our model relative to the Heston and Nandi (2000) affine GARCH option pricing model we report the Implied Volatility Root Mean Squared Error (IVRMSE) measure, defined below:

$$IVRMSE = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(IV - IV^{\mathrm{mkt}})^2} \times 100.$$

Here, the IV and IV^{mkt} represent the Black-Scholes implied volatilities corresponding to the prices associated to the proposed model and to the observed market quotes, respectively.

In Table 4 we present the IVRMSEs for the case in which no market price of factor risk is considered, that is, $\eta_2 = 0$ and our estimates are based solely on the historical return data, while in Table 5 we illustrate the case when η_2 is calibrated using option prices. Both tables contain three modules structured as follows: the"In-Sample Error" box reports the average IVRMSE for each Wednesday of Sample A with parameters calibrated based on option quotes from the same day; the "Next Day Pricing Error" box reports the average IVRMSE for each Thursday of Sample B with parameters calibrated based on option quotes from the previous day; in the box called "Next Week Pricing Error" we report the average IVRMSE for each Wednesday from Sample A using the parameters calibrated based on option quotes from the preceding Wednesday.

The results from Tables 4 and 5 indicate that our ARSV model consistently outperforms the HNGARCH model for the in and out-of-sample exercises under both risk-neutral measure considered. For example, by examining Tables 4 for the in-sample-scenario, we notice that the overall IVRMSE for the ARSV model is 2.953, while the one for the HNGARCH process is 4.040.The improvement for options with maturities below 180 days ranges from around 15% to 40%, the effect being more pronounced for short term options. Furthermore, the ARSV model outperforms the HNGARCH for all groups of maturities considered, with an overall improvement of around 28% for deep ITM options, 25% for ATM options and 21% for deep OTM contracts. For the out-of-sample analysis, we notice a similar behaviour for the pricing errors. When pricing with the ARSV model vs the HNGARCH benchmark, the overall IVRMSE is reduced by around 28% for both the next day and the next week pricing exercises. The only scenario when the HNGARCH outperforms the ARSV counterpart is for very long deep OTM options. When the latent factor risk is priced, our proposed model outperforms the GARCH benchmark in a similar way for the in-sample case, but the out-of-sample overall pricing differences are slightly smaller. For instance, for the "Next Day Pricing Error" the overall improvement is of 18%, while for the "Next Week Pricing Error" is of 12%. Finally, we document in Table 5 the same patterns regarding the behaviour of the IVRMSE with respect to the moneyess and maturity classes as in Table 4.

4.4.2 Hedging performance

The hedging portfolio are constructed using the locally risk minimization ratios described in Section 4.3 and are rebalanced on a weekly basis. We use the following normalized hedging error (NHS) to assess the performance of our autoregressive model relative to the HNGARCH model and the Black-Scholes (B-S) model:[10]

$$\mathrm{NHE}(\xi) := \frac{\left| H(S_T) - V_0 - \sum_{i=0}^{K} \widehat{\xi}_{t_{i+1}} \cdot \left(S_{t_{i+1}} - S_{t_i}\right) \right|}{V_0}$$

Here, $H(S_T)$ is the option payoff at expiration, V_0 is the option price at time 0, S_t is the observed value of the underlying at time t, and $\{t_0 = 0, t_1, \ldots, t_K\}$ represent the set of rebalancing dates during the lifetime of the option. The ratios $\widehat{\xi}_t$ are computed using the formulas (4.7)-(4.9). The results are reported in Table 6. As in the option pricing case, we notice that hedging with the ARSV dynamics is preferred to hedging using the corresponding HNGARCH and B-S ones. The overall NHE for the ARSV model is of 0.491 compared to an NHE of 0.629 for the HNGARCH and 0.758 for the B-S models. A closer look at Table 6 reveals that the ARSV outperforms both models for all 20 groups of maturity and moneyness considered. The largest improvement of 48% over the HNGARCH counterpart is observed for OTM options with maturity $T < 30$, this difference becoming smaller for ITM options.

5 Conclusions

In this paper we propose a one-factor autoregressive SV model for pricing and hedging European style options. Using an exponential affine pricing kernel which contains both equity and latent factor risk preferences, we derive the risk-neutral generating functions for the asset returns and for the factor process. The change of measure preserves the conditional distribution of the factor model, but not that of the asset return process.

We provide a detailed empirical analysis to assess the pricing and hedging performance of our model using both historical returns and option quotes written on the S&P 500 index. The implementation is based on a sequential type algorithm where the model parameters are estimated first using the h-likelihood method and then pricing kernel parameters are calibrated to the observed market prices. For the hedging part, we construct two-instrument portfolios formed with a riskless asset and the underlying and we compute the hedging ratios using a local risk minimization criterion.

Our numerical results indicate that our autoregressive SV model consistently outperform the Heston and Nandi GARCH option pricing model for almost all classes of moneyness and maturity considered. The one week out-of-sample improvements are substantial of around 28% in the case of a zero market price of risk and 12% when the factor risk preference are included in the pricing kernel. The only situation when the HNGARCH model outperforms the ARSV is for very deep OTM long maturity options. The hedging results also support the findings from the pricing exercise that the ARSV model is superior to the HNGARCH counterpart.

Our framework can be further extended to accommodate for multi-factor dynamics and non-Gaussian factor processes, and a more detailed comparison with more complex GARCH dynamics and other

[10] The Black-Scholes portfolios are constructed using the usual BS delta hedging formulas.

discrete SV models has to be performed.

6 Appendix

6.1 Proof of Proposition 2.1

First, we express the the $\mathcal{F}_{t-1}$-conditional bivariate cumulant generating function (c.g.f.) of y_t and f_t under P in terms of the corresponding conditional c.g.f. of f_t and f_t^2:

$$\begin{aligned}
C^P_{(y_t,f_t)}(z_1,z_2|\mathcal{F}_{t-1}) &:= \log \mathrm{E}^P\left[\exp\left(z_1 y_t + z_2 f_t\right)|\mathcal{F}_{t-1}\right] = \log \mathrm{E}^P\left[\exp\left(z_1\left(\mu+\lambda f_t^2\right)+z_1 f_t\epsilon_t + z_2 f_t\right)|\mathcal{F}_{t-1}\right] \\
&= \log \mathrm{E}^P\left[\mathrm{E}^P\left[\exp\left(z_1\left(\mu+\lambda f_t^2\right)+z_1 f_t\epsilon_t + z_2 f_t\right)|\mathcal{G}_t\right]|\mathcal{F}_{t-1}\right] \\
&= \log \mathrm{E}^P\left[\exp\left(z_1\left(\mu+\lambda f_t^2\right)+ z_2 f_t\right)\mathrm{E}^P\left[\exp\left(z_1 f_t\epsilon_t\right)|\mathcal{G}_t\right]|\mathcal{F}_{t-1}\right] \\
&= z_1\mu + C^P_{(f_t,f_t^2)}\left(z_2, z_1\lambda + \frac{1}{2}z_1^2|\mathcal{F}_{t-1}\right) \qquad (6.1)
\end{aligned}$$

Since $f_t|_{\mathcal{F}_{t-1}} \overset{P}{\sim} \mathbf{N}\left(m_{t-1},\sigma_\omega^2\right)$, with $m_t = \gamma + \phi f_t$, it follows that f_t^2 has an $\mathcal{F}_{t-1}$-conditional scaled non-central Chi-Square distribution under P with 1 degree of freedom, scaling parameter σ_ω^2 and noncentrality parameter $m_{t-1}^2/\sigma_\omega^2$:

$$f_t^2|_{\mathcal{F}_{t-1}} \overset{P}{\sim} \mathbf{sn}\chi_{\mathbf{1}}^{\mathbf{2}}\left(\frac{m_{t-1}^2}{\sigma_\omega^2},\sigma_\omega^2\right).$$

The conditional c.g.f. of this distribution is given by:

$$C^P_{f_t^2}(z|\mathcal{F}_{t-1}) = -\frac{1}{2}\log\left(1-2z\sigma_\omega^2\right) + \frac{z m_{t-1}^2}{1-2z\sigma_\omega^2}, \quad 2z\sigma_\omega^2 < 1.$$

Finally, the bivariate conditional c.g.f. of f_t and f_t^2 is given by:

$$C^P_{(f_t,f_t^2)}(z_1,z_2|\mathcal{F}_{t-1}) = -\frac{1}{2}\log\left(1-2z_2\sigma_\omega^2\right) - \frac{m_{t-1}^2}{2\sigma_\omega^2} + \frac{\sigma_\omega^2}{2(1-2z_2\sigma_\omega^2)}\left(z_1 + \frac{m_{t-1}}{\sigma_\omega^2}\right)^2, \quad 2z_2\sigma_\omega^2 < 1.$$

Replacing the above equation into (6.1) and using the notations from (2.6)-(2.7), we find that:

$$C^P_{(y_t,f_t)}(z_1,z_2|\mathcal{F}_{t-1}) = z_1\mu - \frac{1}{2}\log u(z_1) + \frac{v\left(z_2,m_{t-1}\right)}{u(z_1)} - \frac{m_{t-1}^2}{2\sigma_\omega^2}, \quad z_1\left(z_1+2\lambda\right)\sigma_\omega^2 < 1, \ \ z_2 \in \mathbb{R}.$$

This completes the proof. ■

6.2 Proof of Proposition 3.1

We compute the conditional bivariate c.g.f. of (y_t, f_t) under the risk neutral measure Q provided in (3.2) as follows:

$$\begin{aligned}
C^Q_{(y_t,f_t)}(z_1,z_2|\mathcal{F}_{t-1}) &:= \log \mathrm{E}^Q\left[\exp\left(z_1 y_t + z_2 f_t\right)|\mathcal{F}_{t-1}\right] = \log \mathrm{E}^P\left[\exp\left(z_1 y_t + z_2 f_t\right)N_t|\mathcal{F}_{t-1}\right] \\
&= C^P_{(y_t,f_t)}\left(z_1+\eta_{1t}, z_2+\eta_{2t}|\mathcal{F}_{t-1}\right) - C^P_{(y_t,f_t)}\left(\eta_{1t},\eta_{2t}|\mathcal{F}_{t-1}\right) \\
&= z_1\mu - \frac{1}{2}\log\frac{u\left(z_1+\eta_{1t}\right)}{u\left(\eta_{1t}\right)} + \frac{v\left(z_2+\eta_{2t},m_{t-1}\right)}{u\left(z_1+\eta_{1t}\right)} - \frac{v\left(\eta_{2t},m_{t-1}\right)}{u\left(\eta_{1t}\right)}. \quad ■
\end{aligned}$$

6.3 Computation of the gradient and the Hessian of the h-loglikelihood function

First, we provide the expressions for the T components of the gradient $\nabla_{\mathbf{f}} h(\mathbf{y}; \mathbf{f}, \boldsymbol{\theta})$ of the h-likelihood (4.1). A straightforward computation yields for $t \leq T-1$:

$$\frac{\partial h(\mathbf{y}; \mathbf{f}, \boldsymbol{\theta})}{\partial f_t} = -\lambda^2 f_t - \frac{1}{f_t} + \frac{(\mu - y_t)^2}{f_t^3} + \frac{\phi\left(f_{t+1} + f_{t-1}\right) + \gamma(1-\phi) - \left(1+\phi^2\right) f_t}{\sigma_\omega^2}, \tag{6.2}$$

and for $t = T$

$$\frac{\partial h(\mathbf{y}; \mathbf{f}, \boldsymbol{\theta})}{\partial f_T} = -\lambda^2 f_T - \frac{1}{f_T} + \frac{(\mu - y_T)^2}{f_T^3} - \frac{f_T - \gamma - \phi f_{T-1}}{\sigma_\omega^2}. \tag{6.3}$$

Regarding the negative Hessian $H\left(\mathbf{y}; \mathbf{f}, \boldsymbol{\theta}\right)$ of the h-loglikelihood, it is a banded matrix of dimension $T \times T$ whose (i,j) elements H_{ij} are defined as $H_{ij} := -\dfrac{\partial h^2(\mathbf{y}; \mathbf{f}, \boldsymbol{\theta})}{\partial f_i \partial f_j}$ and for any $i, j \leq T-1$ are determined by the following relations

$$H_{ij} = \begin{cases} \lambda^2 - \dfrac{1}{f_i^2} + 3\dfrac{(\mu - y_i)^2}{f_i^4} + \dfrac{1+\phi^2}{\sigma_\omega^2}, & \text{if } \ i = j, \\ -\dfrac{\phi}{\sigma_\omega^2}, & \text{if } \ |i-j| = 1, \\ 0, & \text{otherwise}, \end{cases} \tag{6.4}$$

which is easy to verify. Additionally, $H_{TT} = \lambda^2 - \dfrac{1}{f_T^2} + 3\dfrac{(\mu - y_T)^2}{f_T^4} + \dfrac{1}{\sigma_\omega^2}$.

6.4 Forecasted and filtered latent factor values

First, we denote by $f_{t|t-1} := \mathrm{E}\left[f_t | \mathcal{F}_{t-1}\right]$ and by $f_{t|t} := \mathrm{E}\left[f_t | \mathcal{F}_t\right]$ the forecasted (predicted) and filtered (updated) values of the latent factor. Let $\mathcal{F}_t^* = \sigma\left(y_s, f_{s|s}; s \leq t\right)$ be the sigma-algebra generated by both the return and the filtered factor processes, and let $\mathcal{G}_t^* = \sigma\left(y_s, f_{s|s}, f_t; s \leq t-1\right) = \mathcal{F}_{t-1}^* \bigcup \{f_t\}$ be its corresponding augmented filtration. In this notation the following relations hold

$$f_{t|t-1} = \underset{f_t \in \mathbb{R}}{\arg\max}\ g(f_t | \mathcal{F}_{t-1}^*), \tag{6.5}$$

$$f_{t|t} = \underset{f_t \in \mathbb{R}}{\arg\max}\ g(f_t | \mathcal{F}_{t-1}^* \cup \{y_t\}) = \underset{f_t \in \mathbb{R}}{\arg\max}\ g(y_t | \mathcal{F}_{t-1}^* \cup \{f_t\}) g(f_t | \mathcal{F}_{t-1}^*) = \underset{f_t \in \mathbb{R}}{\arg\max}\ g(y_t | \mathcal{G}_t^*) g(f_t | \mathcal{F}_{t-1}^*), \tag{6.6}$$

where we can use the conditional distribution properties provided in (2.3)-(2.4). We notice that both the return process and the latent factor process are conditionally Gaussian distributed with respect to $\mathcal{G}_t^*$ and $\mathcal{F}_{t-1}^*$, respectively, that is

$$y_t|_{\mathcal{G}_t^*} \sim \mathbf{N}\left(\mu + \lambda f_t^2, f_t^2\right) \ \text{ and } \ f_t|_{\mathcal{F}_{t-1}^*} \sim \mathbf{N}\left(\gamma + \phi f_{t-1|t-1}, \sigma_\omega^2\right). \tag{6.7}$$

We can hence rewrite the optimization problems (6.5)-(6.6) in the following equivalent form

$$
\begin{aligned}
f_{t|t-1} &= \operatorname*{arg\,min}_{f_t \in \mathbb{R}} \left(f_t - \gamma - \phi f_{t-1|t-1}\right)^2 = \gamma + \phi f_{t-1|t-1}, && (6.8)\\
f_{t|t} &= \operatorname*{arg\,min}_{f_t \in \mathbb{R}} u(f_t), \;\; \text{with} \;\; u(f_t) := -\frac{1}{2}\left(\log(f_t^2) + \frac{(y_t - \mu - \lambda f_t^2)^2}{f_t} + \frac{(f_t - \gamma - \phi f_{t-1|t-1})^2}{\sigma_\omega^2}\right). && (6.9)
\end{aligned}
$$

Again, the extrema in the optimization problem in (6.9) can be found solving the score equation

$$
\frac{du(f_t)}{df_t} = -\lambda^2 f_t - \frac{1}{f_t} + \frac{(\mu - y_t)^2}{f_t^3} - \frac{f_t}{\sigma_\omega^2} + \frac{\gamma + \phi f_{t-1|t-1}}{\sigma_\omega^2} = 0
$$

with respect to f_t. Expressions in (6.8) and (6.9) provide the forecasted and the filtered latent factor values, respectively, as required.

References

Babaoglu, K., Christoffersen, P. F., Heston, S. and Jacobs, K. (2014). Option valuation with volatility components, fat tails, and non-linear pricing kernels.

Badescu, A., Cui, Z. and Ortega, J.-P. (2015). Non-affine GARCH option pricing models, variance dependent kernels, and diffusion limits.

Badescu, A., Elliott, R. J. and Ortega, J.-P. (2014). Quadratic hedging schemes for non-Gaussian GARCH models. Journal of Economic Dynamics and Control *42*, 13–32.

Badescu, A. M. and Kulperger, R. J. (2008). GARCH option pricing: A semiparametric approach. Insurance: Mathematics and Economics *43*, 69–84.

Bakshi, G., Cao, C. and Chen, Z. (1997). Empirical performance of alternative option pricing models. The Journal of Finance *52*, 2003–2049.

Bertholon, H., Monfort, A. and Pegoraro, F. (2008). Econometric asset pricing modeling. Journal of Financial Econometrics *6*, 407–458.

Black, F. and Scholes, M. (1973). The pricing of options and corporate liabilities. Journal of Political Economy *81*, 637–659.

Bollerslev, T. (1986). Generalized autoregressive conditional heteroskedasticity. Journal of Econometrics *31*, 307–327.

Bormetti, C., Corsi, F. and Majewski, A. A. (2015). Term structure of variance risk premium in multi-component GARCH models.

Chorro, C., Guegan, D. and Ielpo, F. (2012). Option pricing for GARCH-type models with generalized hyperbolic innovations. Quantitative Finance *12*, 1079–1094.

Christoffersen, P., Feunou, B., Jacobs, K. and Meddahi, N. (2014). The economic value of realized volatility: using high-frequency returns for option valuation. Journal of Financial and Quantitative Analysis *49*, 663–697.

Christoffersen, P., Heston, S. and Jacobs, K. (2006). Option valuation with conditional skewness. Journal of Econometrics *131*, 253–284.

Christoffersen, P., Heston, S. L. and Jacobs, K. (2013). Capturing option anomalies with a variance-dependent pricing kernel. Review of Financial Studies *26*, 1963–2006.

Christoffersen, P. F., Jacobs, K., Ornthanalai, C. and Wang, Y. (2008). Option valuation with long-run and short-run volatility components. Journal of Financial Economics *90*, 272–297.

Corsi, F., Fusari, N. and La Vecchia, D. (2013). Realizing smiles: Options pricing with realized volatility. Journal of Financial Economics *107*, 284–304.

Darolles, S., Gourieroux, C. and Jasiak, J. (2006). Structural laplace transform and compound autoregressive models. Journal of Time Series Analysis *27*, 477–503.

del Castillo, J. and Lee, Y. (2008). GLM-methods for volatility models. Stat. Model. *8*, 263–283.

Duan, J.-C. (1995). The GARCH option pricing model. Mathematical Finance *5*, 13–32.

Eichler, A., Leobacher, G. and Zellinger, H. (2011). Calibration of financial models using quasi-Monte Carlo. Monte Carlo Methods and Applications *17*, 99–131.

Engle, R. F. (1982). Autoregressive conditional heteroscedasticity with estimates of the variance of United Kingdom inflation. Econometrica *50*, 987–1007.

Feunou, B. and Tédongap, R. (2012). A stochastic volatility model with conditional skewness. Journal of Business & Economic Statistics *30*, 576–591.

Föllmer, H. and Schied, A. (2002). Stochastic Finance, vol. 27, of de Gruyter Studies in Mathematics. Walter de Gruyter and Co., Berlin.

Föllmer, H. and Sondermann, D. (1986). Hedging of nonredundant contingent claims. In Contributions to mathematical economics pp. 205–223. North-Holland Amsterdam.

Gouriéroux, C. and Jasiak, J. (2006). Autoregressive Gamma Processes. Journal of Forecasting *25*, 129–152.

Hansen, P. R., Huang, Z. and Shek, H. H. (2011). Realized GARCH: a joint model for returns and realized measures of volatility. Journal of Applied Econometrics *27*, 877–906.

Harvey, A. C., Ruiz, E. and Shephard, N. (1994). Multivariate stochastic variance models. Review of Economic Studies *61*, 247–264.

Heston, S. L. and Nandi, S. (2000). A closed-form GARCH option valuation model. Review of Financial Studies *13*, 585–625.

Lee, Y. and Nelder, J. A. (1996). Hierarchical generalized linear models. J. Roy. Statist. Soc. Ser. B *58*, 619–678.

Lee, Y. and Nelder, J. A. (2006). Double hierarchical generalized linear models. J. Roy. Statist. Soc. Ser. C *55*, 139–185.

Lim, J., Woojoo, L., Lee, Y. and del Castillo, J. (2011). The hierarchical-likelihood approach to autoregressive stochastic volatility models. Computational Statistics and Data Analysis *55*, 248–260.

Majewski, A. A., Bormetti, G. and Corsi, F. (2015). Smile from the past: A general option pricing framework with multiple volatility and leverage components. Journal of Econometrics *187*, 521–531.

Pollock, D. (2003). Recursive estimation in econometrics. Computational Statistics & Data Analysis *44*, 37–75.

Renault, E. and Khrapov, S. (2014). Affine option pricing model in discrete time.

Stentoft, L. (2008). Option pricing using realized volatility.

Taylor, S. J. (1986). Modelling Financial Time Series. John Wiley & Sons, Chichester.

Taylor, S. J. (2005). Asset Price Dynamics, Volatility, and Prediction. Princeton University Press, Princeton.

Trolle, A. B. and Schwartz, E. S. (2009). Unspanned stochastic volatility and the pricing of commodity derivatives. Review of Financial Studies *22*, 4423–4461.

BASIC FEATURES OF THE OPTION PRICING (CALLS) DATASET (WEDNESDAYS)

		Moneyness S_0/K						Across
	Maturities	[0.900, 0.950]	[0.950, 0.975]	[0.975, 1.000]	[1.000, 1.025]	[1.025, 1.050]	[1.050, 1.100]	**Moneyness**
Number of Contracts	$T < 30$	471	403	414	232	63	28	1611
	$30 \leq T < 80$	811	716	722	541	111	54	2955
	$80 \leq T < 180$	373	211	209	192	46	32	1063
	$180 \leq T \leq 250$	128	62	83	99	27	19	418
Across Maturities		1783	1392	1428	1064	247	133	6047
Average Prices	$T < 30$	0.715	3.728	14.244	32.236	62.791	102.548	13.682
	$30 \leq T < 80$	3.401	10.568	25.259	42.978	69.068	108.476	22.111
	$80 \leq T < 180$	15.927	32.498	51.721	68.634	92.834	130.495	42.551
	$180 \leq T \leq 250$	32.932	56.488	75.323	92.194	110.254	140.487	68.762
Across Maturities		7.432	13.957	28.848	49.845	76.395	117.099	26.683
Average Implied Volatilities	$T < 30$	0.128	0.118	0.129	0.143	0.169	0.192	0.131
	$30 \leq T < 80$	0.124	0.125	0.139	0.152	0.165	0.180	0.136
	$80 \leq T < 180$	0.138	0.149	0.161	0.165	0.179	0.198	0.153
	$180 \leq T \leq 250$	0.154	0.167	0.175	0.181	0.184	0.193	0.170
Across Maturities		0.130	0.129	0.142	0.155	0.171	0.189	0.140

Table 1: Basic features of the Sample A option dataset (Wednesdays). Prices in this dataset correspond to the period January 1st, 2012–December 31st, 2013.

BASIC FEATURES OF THE OPTION PRICING (CALLS) DATASET (THURSDAYS)

		Moneyness S_0/K						Across
	Maturities	[0.900, 0.950]	[0.950, 0.975]	[0.975, 1.000]	[1.000, 1.025]	[1.025, 1.050]	[1.050, 1.100]	**Moneyness**
Number of Contracts	$T < 30$	332	403	436	262	81	38	1552
	$30 \leq T < 80$	913	769	778	584	110	63	3217
	$80 \leq T < 180$	404	213	219	191	50	40	1117
	$180 \leq T \leq 250$	129	62	100	96	29	13	429
Across Maturities		1778	1447	1533	1133	270	154	6315
Average Prices	$T < 30$	0.851	3.762	13.742	32.452	60.877	98.193	16.079
	$30 \leq T < 80$	3.501	10.516	24.820	43.546	69.164	107.068	21.877
	$80 \leq T < 180$	16.502	32.378	50.652	70.158	92.345	129.480	42.840
	$180 \leq T \leq 250$	33.122	54.921	75.785	93.668	112.984	146.419	68.598
Across Maturities		8.109	13.756	28.684	49.714	75.677	114.021	27.334
Average Implied Volatilities	$T < 30$	0.128	0.118	0.127	0.143	0.161	0.196	0.131
	$30 \leq T < 80$	0.123	0.125	0.138	0.153	0.164	0.190	0.135
	$80 \leq T < 180$	0.139	0.148	0.159	0.170	0.174	0.196	0.154
	$180 \leq T \leq 250$	0.152	0.162	0.173	0.180	0.184	0.196	0.168
Across Maturities		0.130	0.128	0.140	0.156	0.167	0.194	0.140

Table 2: Basic features of the Sample B option dataset (Thursdays). Prices in this dataset correspond to the period January 1st, 2004–December 31st, 2013.

BASIC FEATURES OF THE OPTION HEDGING (PUTS) DATASET							
		Moneyness S_0/K					**Across**
	Maturities	$[0.950, 0.975]$	$[0.975, 1.000]$	$[1.000, 1.025]$	$[1.025, 1.050]$	$[1.050, 1.100]$	**Moneyness**
Number of Contracts	$T < \mathbf{30}$	18	67	225	210	346	866
	$\mathbf{30} \leq T < \mathbf{80}$	17	114	422	298	495	1346
	$\mathbf{80} \leq T < \mathbf{180}$	3	51	130	80	149	413
	$\mathbf{180} \leq T \leq \mathbf{250}$	0	12	41	21	29	103
Across Maturities		38	244	818	609	1019	2728
Average Prices	$T < \mathbf{30}$	58.933	33.557	19.404	10.318	5.034	13.376
	$\mathbf{30} \leq T < \mathbf{80}$	73.138	46.383	34.163	23.088	14.194	25.895
	$\mathbf{80} \leq T < \mathbf{180}$	98.767	79.584	65.802	50.420	38.948	55.076
	$\mathbf{180} \leq T \leq \mathbf{250}$	——	108.304	97.965	81.412	69.350	87.738
Across Maturities		68.433	52.846	38.329	24.286	16.273	28.673
Average Implied Volatilities	$T < \mathbf{30}$	0.133	0.145	0.148	0.168	0.198	0.172
	$\mathbf{30} \leq T < \mathbf{80}$	0.129	0.149	0.155	0.172	0.190	0.171
	$\mathbf{80} \leq T < \mathbf{180}$	0.154	0.167	0.168	0.180	0.198	0.181
	$\mathbf{180} \leq T \leq \mathbf{250}$	——	0.175	0.176	0.178	0.205	0.184
Across Maturities		0.133	0.153	0.156	0.172	0.194	0.173

Table 3: Basic features of the Sample C option dataset (Wednesdays). Prices in this dataset correspond to the period January 1st, 2012–December 31st, 2013.

OUT OF SAMPLE PRICING ERRORS (IVRMSE in percentage) WITH SEQUENTIALLY ESTIMATED PARAMETERS AND $\eta_2 = 0$															
		Moneyness S_0/K												Across	
	Maturities	$[0.900, 0.950]$		$[0.950, 0.975]$		$[0.975, 1.000]$		$[1.000, 1.025]$		$[1.025, 1.050]$		$[1.050, 1.100]$		Moneyness	
	Model	HNGARCH	ARSV	HNGARCH	ARSV	HNGARCH	ARSV	HNGARCH	ARSV	HNGARCH	ARSV	HNGARCH	ARSV	HNGARCH	ARSV
In-Sample Error	$T < 30$	2.991	1.787	3.279	2.301	4.062	2.482	4.711	2.575	5.069	3.408	8.425	3.535	3.871	2.338
	$30 \leq T < 80$	3.073	2.050	3.732	2.249	4.287	2.656	4.277	3.126	5.777	3.616	5.832	4.260	3.961	2.592
	$80 \leq T < 180$	3.506	2.779	4.371	3.187	4.205	4.170	4.691	3.803	5.369	5.182	5.343	5.590	4.203	3.587
	$180 \leq T \leq 250$	3.974	3.812	5.553	4.596	4.698	5.304	4.667	5.414	6.198	6.477	4.937	7.048	4.747	5.002
Across Maturities		3.219	2.325	3.814	2.573	4.236	3.086	4.488	3.429	5.576	4.281	6.262	4.942	4.040	2.953
Next Day Pricing Error	$T < 30$	2.994	1.862	2.604	2.218	3.788	2.323	4.474	2.723	5.586	3.107	7.324	3.304	3.753	2.359
	$30 \leq T < 80$	3.272	1.983	3.441	2.207	4.384	2.467	4.034	3.179	5.885	3.403	5.048	4.042	3.894	2.510
	$80 \leq T < 180$	3.353	2.854	4.051	3.317	4.670	3.699	4.059	4.234	4.989	4.349	5.483	5.080	4.062	3.544
	$180 \leq T \leq 250$	3.876	3.841	5.261	4.114	5.057	4.513	5.027	5.488	3.638	5.929	4.861	7.754	4.662	4.749
Across Maturities		3.290	2.368	3.437	2.515	4.318	2.816	4.235	3.540	5.434	3.852	5.794	4.606	3.948	2.891
Next Week Pricing Error	$T < 30$	3.062	1.785	3.096	2.340	3.827	2.572	4.801	2.682	5.990	3.524	11.046	3.752	3.963	2.402
	$30 \leq T < 80$	3.105	2.062	3.807	2.291	4.347	2.690	4.464	3.046	5.437	3.704	6.208	4.147	4.028	2.593
	$80 \leq T < 180$	3.412	2.827	3.998	3.227	3.860	4.179	5.154	3.769	5.478	5.247	5.800	5.746	4.158	3.613
	$180 \leq T \leq 250$	4.588	3.586	4.522	4.693	4.998	5.240	5.218	5.158	4.293	6.570	4.682	7.307	4.808	4.913
Across Maturities		3.283	2.313	3.679	2.614	4.172	3.115	4.744	3.360	5.498	4.349	7.244	5.051	4.091	2.958

Table 4: Option pricing results for the autoregressive SV (ARSV) model and the Heston and Nandi GARCH (HNGARCH) model computed using the conditional Esscher transform which corresponds to a zero price of factor risk, $\eta_2 = 0$. The parameter values are computed based solely on the asset return daily data and are reestimated every four weeks. The"In-Sample Error" box reports the average IVRMSE for each Wednesday. The "Next Day Pricing Error" box reports the average IVRMSE committed each Thursday using the models whose parameters have been eventually estimated the preceding day. The module marked "Next Week Pricing Error" reports the average IVRMSE for each Wednesday using the models whose parameters have been eventually estimated the Wednesday of the preceding week.

OUT OF SAMPLE PRICING ERRORS (IVRMSE in percentage) WITH SEQUENTIALLY ESTIMATED PARAMETERS AND η_2 OBTAINED BY CALIBRATION															
		Moneyness S_0/K												Across	
	Maturities	$[0.900, 0.950]$		$[0.950, 0.975]$		$[0.975, 1.000]$		$[1.000, 1.025]$		$[1.025, 1.050]$		$[1.050, 1.100]$		Moneyness	
	Model	HNGARCH	ARSV	HNGARCH	ARSV	HNGARCH	ARSV	HNGARCH	ARSV	HNGARCH	ARSV	HNGARCH	ARSV	HNGARCH	ARSV
In-Sample Error	$T < 30$	2.748	1.548	2.982	1.828	3.728	2.151	4.102	2.992	4.649	4.359	8.394	4.640	3.535	2.265
	$30 \leq T < 80$	2.956	1.777	3.639	1.934	4.099	2.568	4.010	3.362	5.353	4.112	5.508	4.527	3.781	2.543
	$80 \leq T < 180$	3.397	2.676	4.255	3.228	4.041	4.236	4.592	4.110	5.372	5.306	5.621	5.391	4.107	3.641
	$180 \leq T \leq 250$	3.922	3.822	5.427	4.527	4.626	5.312	4.560	5.438	6.121	5.913	5.015	7.355	4.669	4.978
Across Maturities		3.079	2.146	3.666	2.322	4.019	2.988	4.194	3.675	5.274	4.632	6.200	5.232	3.845	2.926
Next Day Pricing Error	$T < 30$	3.004	1.798	2.570	2.231	3.622	2.853	4.066	3.513	5.663	4.606	7.777	7.122	3.654	2.959
	$30 \leq T < 80$	3.128	1.804	3.281	2.100	4.238	2.634	3.774	3.423	5.976	4.197	4.735	4.627	3.735	2.609
	$80 \leq T < 180$	3.304	2.902	3.965	3.602	4.668	4.002	4.102	4.759	5.458	5.043	5.677	5.515	4.075	3.842
	$180 \leq T \leq 250$	3.876	4.183	5.218	4.218	5.116	4.976	5.066	5.881	3.453	6.150	4.667	7.828	4.663	5.073
Across Maturities		3.207	2.341	3.328	2.536	4.206	3.125	4.023	3.959	5.568	4.721	5.870	5.875	3.849	3.163
Next Week Pricing Error	$T < 30$	2.990	1.646	2.918	2.013	3.566	2.776	4.443	4.213	5.437	6.337	10.791	7.724	3.741	2.989
	$30 \leq T < 80$	3.162	1.939	3.725	2.293	4.187	2.964	4.293	3.752	5.165	4.253	6.291	5.119	3.934	2.858
	$80 \leq T < 180$	3.329	3.091	3.934	3.811	3.771	4.918	5.108	5.487	5.595	9.447	5.753	6.303	4.100	4.628
	$180 \leq T \leq 250$	4.593	4.223	4.560	5.245	4.972	5.590	5.172	6.315	4.705	7.156	5.011	8.774	4.835	5.654
Across Maturities		3.272	2.381	3.583	2.700	4.005	3.464	4.570	4.492	5.280	6.385	7.214	6.607	3.980	3.509

Table 5: Option pricing results for the autoregressive SV (ARSV) model and the Heston and Nandi GARCH (HNGARCH) model computed using the factor dependent pricing kernel which corresponds to calibrated prices of factor risk η_2. The model parameters are computed based on daily asset returns and are re-estimated every four weeks and the pricing kernel parameter η_2 is estimated based on the observed option quotes. The"In-Sample Error" box reports the average IVRMSE for each Wednesday at the time of pricing the options that have been used to optimize the options likelihood with respect to η_2 the very same Wednesday. The "Next Day Pricing Error" box reports the average IVRMSE committed each Thursday using the models whose parameters have been estimated the preceding day. The module marked "Next Week Pricing Error" reports the average IVRMSE for each Wednesday using the models whose parameters have been estimated the Wednesday of the preceding week..

RESULTS FOR THE 2012-2013 HEDGING EXERCISE							
		Moneyness S_0/K					**Across**
	Maturities	$[0.950, 0.975]$	$[0.975, 1.000]$	$[1.000, 1.025]$	$[1.025, 1.050]$	$[1.050, 1.100]$	**Moneyness**
NHSE Black-Scholes	$T < \mathbf{30}$	0.745	0.945	1.108	1.113	0.710	0.930
	$\mathbf{30} \leq T < \mathbf{80}$	0.731	0.753	0.806	0.736	0.596	0.699
	$\mathbf{80} \leq T < \mathbf{180}$	0.629	0.651	0.666	0.614	0.502	0.586
	$\mathbf{180} \leq T \leq \mathbf{250}$	——	0.618	0.662	0.649	0.510	0.581
Across Maturities		0.730	0.778	0.860	0.847	0.618	0.758
NHSE HNGARCH	$T < \mathbf{30}$	0.757	0.959	1.107	1.131	0.770	0.945
	$\mathbf{30} \leq T < \mathbf{80}$	0.682	0.716	0.743	0.684	0.527	0.670
	$\mathbf{80} \leq T < \mathbf{180}$	0.542	0.520	0.474	0.449	0.346	0.466
	$\mathbf{180} \leq T \leq \mathbf{250}$	——	0.446	0.399	0.345	0.348	0.384
Across Maturities		0.660	0.660	0.681	0.652	0.498	0.629
NHSE ARSV	$T < \mathbf{30}$	0.673	0.807	0.925	0.868	0.396	0.734
	$\mathbf{30} \leq T < \mathbf{80}$	0.642	0.626	0.621	0.477	0.312	0.535
	$\mathbf{80} \leq T < \mathbf{180}$	0.473	0.426	0.384	0.316	0.185	0.357
	$\mathbf{180} \leq T \leq \mathbf{250}$	——	0.400	0.318	0.262	0.228	0.302
Across Maturities		0.596	0.565	0.562	0.481	0.280	0.491

Table 6: Average normalized hedging square errors (NHSE) associated with portfolios constructed based on the autoregressive SV model (ARSV), the Heston-Nandi GARCH (HNGARCH) and the Black-Scholes (BS) model; the hedging strategies for the ARSV and HN are computed using the local risk minimization criteria, and the BS ones are based on the Black-Scholes delta hedging formula. Each entry in the table has been computed by averaging the normalized hedging errors committed when handling the options contained in the corresponding moneyness-time to maturity bin.