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Asymptotic and Lyapunov stability of constrained and Poisson equilibria

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Abstract

This paper includes results centered around three topics, all of them related with the nonlinear stability of equilibria in constrained dynamical systems. First, we prove an energy-Casimir type sufficient condition for stability that uses functions that are not necessarily conserved by the flow and that takes into account the asymptotically stable behavior that may occur in certain constrained systems, such as Poisson and Leibniz dynamical systems. Second, this method is specifically adapted to Poisson systems obtained via a reduction procedure and we show in examples that the kind of stability that we propose is appropriate when dealing with the stability of the equilibria of some constrained mechanical systems. Finally, we discuss two situations in which the use of continuous Casimir functions in stability studies is equivalent to the topological stability methods introduced by Patrick et al. (Arch. Rational Mech. Anal., 2004, preprint arXiv:math.DS/0201239v1, to appear).

Keywords: Stability; Hamiltonian systems; Poisson dynamical systems

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1. Introduction

The use of the conserved quantities of a Hamiltonian flow in the study of the stability of its solutions is a venerable topic that goes back to Lagrange and Dirichlet. In the past decades these ideas have been adapted to various setups: equilibria in Poisson systems [A66,Hoal85,Paal04], relative equilibria [Pa92,LS98,Or98,OrRa99,Paal04] and periodic and relative periodic orbits [OrRa99a,OrRa99b] of symmetric Hamiltonian systems, relative equilibria of symmetric Lagrangian systems [SLM91], and symmetric nonholonomically constrained mechanical systems [Zeal98], to list a few. All these results provide sufficient conditions for the solution in question to be stable.

In this paper, we will focus on the stability of the equilibria of constrained dynamical systems, that is, vector fields whose flows preserve submanifolds that are naturally defined in the problem as leaves of foliations or level sets of continuous functions (integrals of motion). The presence of such systems is widespread in applications. For example, any Hamiltonian system on a Poisson manifold can be thought of as a constrained system due to the dynamical preservation of its symplectic leaves (these terms are briefly explained later on in this introduction). The main tools that one finds in the literature concerning this case are the energy-Casimir method and the topological stability methods introduced in [Paal04]. The energy-Casimir method consists of finding a combination of conserved quantities by the Hamiltonian flow, typically the energy and the Casimir functions, that exhibits a critical point at the equilibrium with definite Hessian. Since the dynamics of the system is confined to the level sets of this combination and, by the Morse Lemma, in a coordinate chart about the equilibrium these level sets are diffeomorphic to spheres centered at the equilibrium, stability follows. The topological methods in [Paal04] rely on a much more subtle confinement of the flow that takes advantage not only of its conservation laws but also of the topological properties of the foliation of the Poisson manifold by its symplectic leaves.

Energy confinement is a very important tool in the symplectic Hamiltonian context due to the absence of asymptotically stable behavior. Energy methods are, to this day, the only general way to prove stability in more than two degrees of freedom. The conservation of the phase space volume by the flow imposed by Liouville's theorem does not necessarily hold in the Poisson category. The first main result of this paper, contained in Theorem 2.5, adapts the standard energy-Casimir method to constrained dynamical systems. Moreover, its statement combines these conservation properties with the use of functions that are not necessarily conserved by the flow but that can still be used to conclude a certain kind of asymptotic stability via the standard Lyapunov stability theorem. This newly introduced notion of stability implies the standard Lyapunov stability and will be referred to as *weak asymptotic stability*. In the particular case of Poisson dynamical systems the occurrence of asymptotically stable behavior has already been observed in [Mar95,Bl00]. In this specific case Theorem 2.5 improves a previous version of the energy-Casimir method (see [Or98] or Corollary 4.11 in [OrRa99b]) where the conserved quantities confining the flow are also used to shrink the space on which one checks the definiteness of the Hessian. Theorem 2.5 shows that *any* conserved quantity can be used to shrink this space even when that conserved quantity is not involved in the construction of a positive definite Hessian.

Theorem 2.12 is the second main result of this paper. It adapts the stability condition in Theorem 2.5 to equilibria of Poisson systems obtained by a certain reduction procedure that uses ideals in the Poisson algebra of the functions on the manifold. Our interest is twofold. First, there are some mechanical systems with holonomic or nonholonomic constraints that can be described by reducing in this sense a bigger (unconstrained) system. Second, the weakened kind of stability that Theorem 2.12 allows us to conclude, coincides with the physically relevant notion of stability in those situations, that is, the one that describes the system when subjected to perturbations compatible with the constraints. We illustrate this point with a couple of examples in Section 3: a light Chaplygin sleigh on a cylinder and two coupled spinning wheels. Second, there are cases when there are not enough conserved quantities to apply Theorem 2.5 but, nevertheless, the system can be reduced around the equilibrium and then the reduced system has enough conserved quantities to use the theorem. Theorem 2.12 explains the meaning of having this *reduced* kind of stability. In particular, it shows the role of sub-Casimir functions in stability computations.

The last section of the paper is dedicated to the study of the relation between the topological stability methods in [Paal04] with a generalized version of the energy-Casimir method that we propose in the text based on the use of local continuous Casimir functions of the Poisson manifold. To be more explicit, the stability criteria in [Paal04] are stated in terms of a set that, roughly speaking, measures how far the space of symplectic leaves of a Poisson manifold is from being a Hausdorff topological space. The general question that we try to answer is under what circumstances this set can be characterized as the intersection of level sets of local continuous Casimirs. Since this is not true in general, we provide two sufficient conditions that are related to certain idempotency of the set in [Paal04] and to the possibility of separating regular symplectic leaves by using continuous Casimirs. The natural category where these questions are posed is that of generalized foliated manifolds; this is the context in which we have formulated the main results in this section and where we have obtained the Poisson case as a byproduct, considering it as a manifold foliated by its symplectic leaves.

Before we start with the core of the paper we quickly review in a few paragraphs the basic notions and terminology of generalized foliations and Poisson and Leibniz manifolds that we will use throughout the paper. In this paper all manifolds are assumed to be finite dimensional Hausdorff and paracompact. All the vector fields are smooth. The expert can safely skip the rest of this section.

1.1. Poisson systems

Let P be a smooth manifold and let $C^\infty(P)$ be the algebra of smooth functions on P. A Poisson structure on P is a bilinear map $\{\cdot,\cdot\} : C^\infty(P) \times C^\infty(P) \longrightarrow C^\infty(P)$ that defines a Lie algebra structure on $C^\infty(P)$ and that is a derivation on each entry. The derivation property allows us to assign to each function $F \in C^\infty(P)$ a vector field $X_F \in \mathfrak{X}(P)$ via the equality

$$X_H[F] := \{F, H\} \text{ for every } F \in C^\infty(P).$$

The vector field $X_H \in \mathfrak{X}(P)$ is called the *Hamiltonian vector field* associated to the *Hamiltonian* function H. The derivation property of the Poisson bracket also implies that for any two functions $F, G \in C^\infty(P)$, the value of the bracket $\{F, G\}(z)$ at an arbitrary point $z \in P$ depends on F only through $\mathbf{d}F(z)$ which allows us to define a contravariant antisymmetric two-tensor $B \in \Lambda^2(P)$ by

$$B(z)(\alpha_z, \beta_z) = \{F, G\}(z),$$

where $\mathbf{d}F(z) = \alpha_z \in T_z^*P$ and $\mathbf{d}G(z) = \beta_z \in T_z^*P$. This tensor is called the *Poisson tensor* of M. The vector bundle map $B^\sharp : T^*P \to TP$ naturally associated to B is defined by $B(z)(\alpha_z, \beta_z) = \langle \alpha_z, B^\sharp(\beta_z)\rangle$. Its range $\mathcal{E} := B^\sharp(T^*P) \subset TP$ is called the *characteristic distribution* of the Poisson manifold $(P, \{\cdot, \cdot\})$. Its value at $z \in P$ is hence given by $\mathcal{E}_z = \{X_H(z) \mid H \in C^\infty(P)\}$. The distribution $\mathcal{E}$ is a smooth generalized distribution which is always integrable in the sense of Stefan [St74a,St74b] and Sussmann [Su73]. Its maximal integral submanifolds $\{\mathcal{L}\}$ are symplectic and are called the *symplectic leaves* of $(P, \{\cdot, \cdot\})$. The symplectic form $\omega_{\mathcal{L}}$ on the leaf $\mathcal{L}$ is uniquely characterized by the identity

$$\omega_{\mathcal{L}}(z)\,(X_F(z), X_G(z)) := \{F, G\}(z) \quad \text{for any} \quad F, G \in C^\infty(P) \text{ and for any } z \in \mathcal{L}.$$

Since the symplectic leaves of $(P, \{\cdot, \cdot\})$ are the maximal integral leaves of a generalized distribution, they form a *generalized foliation* in the sense of [Daz85]. This implies the existence of a chart $(U, \varphi : U \to \mathbb{R}^m)$ around any point $z \in P$ such that if $\mathcal{L}_z$ is the symplectic leaf containing z then there is a countable subset $A \subset \mathbb{R}^{m-n}$, with $m = \dim P$ and $n = \dim \mathcal{L}_z$, such that

$$\varphi(U \cap \mathcal{L}_z) = \{y \in \varphi(U) \mid (y^{n+1}, \ldots, y^m) \in A\}. \tag{1.1}$$

Such a chart (U, φ) is called a *foliation chart* for the generalized symplectic foliation of P around the point z. A connected component of $U \cap \mathcal{L}_z$ is called a *plaque* of the foliation chart (U, φ). The point z is said to be *regular* if the neighborhood U can be shrunk so that all the leaves that it intersects have all the same dimension. In that case, the plaques coincide with the points of the form $(y^1, \ldots, y^n, y_0^{n+1}, \ldots, y_0^m) \in \varphi(U)$ with $(y_0^{n+1}, \ldots, y_0^m)$ constant. A leaf consisting of regular points is said to be *regular* and *singular* otherwise. The set of regular points of a generalized smooth foliation is open and dense.

Some of the results proved in this paper will be first given in the category of foliated manifolds. The corresponding results in the context of Poisson manifolds are then obtained as corollaries.

1.2. Casimirs, local Casimirs, and first integrals of foliations

A function on a foliated manifold that is constant on the leaves is called a *first integral* of the foliation. When we consider the particular case of a Poisson manifold, the elements in the center of the Poisson algebra $(C^\infty(P), \{\cdot,\cdot\})$, also called the *Casimir* functions, are first integrals of the foliation of P by its symplectic leaves. A *local Casimir* at the point $z \in P$ is a function $C \in C^\infty(U_z)$ for some open neighborhood $U_z \subset P$ of z such that it is a Casimir of the Poisson manifold $(U_z, \{\cdot,\cdot\}_{U_z})$ where the bracket $\{\cdot,\cdot\}_{U_z}$ is the restriction of the bracket $\{\cdot,\cdot\}$ on P to U_z.

In general, nontrivial global Casimir functions may not exist. On the other hand, local Casimirs are always available in the neighborhood of a regular point. Indeed, if we think of the Poisson manifold $(P, \{\cdot,\cdot\})$ as a foliated space by its symplectic leaves, the expression (1.1) allows us to find a chart $(U, \varphi : U \to \mathbb{R}^m)$ around the regular point where the plaques of the symplectic foliation are the points of the form $(y^1, \ldots, y^n, y_0^{n+1}, \ldots, y_0^m) \in \varphi(U)$ with $(y_0^{n+1}, \ldots, y_0^m)$ constant. The functions that depend on the last $m-n$ coordinates are local Casimir functions of $(P, \{\cdot,\cdot\})$ around z.

1.3. Quasi-Poisson submanifolds and sub-Casimirs

An embedded submanifold S of P which is Poisson in its own right and is such that the inclusion $i : S \hookrightarrow P$ is canonical is called a *Poisson submanifold* of P. The Poisson structure on S is uniquely determined by the condition that the inclusion be canonical, that is, there is no other Poisson structure on S relative to which the inclusion is canonical.

It turns out that in this paper we need a slightly weaker condition. An embedded submanifold S of P (without any Poisson structure on it) such that $B^\sharp(s)\left(T_s^*P\right) \subset T_sS$ for any $s \in S$ is called a *quasi-Poisson submanifold* of P. Every Poisson submanifold is quasi-Poisson but the converse is not true. As a corollary to the main theorem in [MaRa86], one can easily conclude that if S is a quasi-Poisson submanifold of P, then there is a unique Poisson structure $\{\cdot,\cdot\}_S$ on S with respect to which the inclusion $S \hookrightarrow P$ is a Poisson map, that is, there is a unique induced Poisson structure on S making it into a Poisson submanifold of P. The Poisson bracket $\{\cdot,\cdot\}_S$ is defined by $\{f,g\}_S(s) := \{F,G\}(s)$ where $F, G \in C^\infty(P)$ are arbitrary local extensions of $f, g \in C^\infty(S)$ around the point $s \in S$; this means that there is an open neighborhood U of s in P such that $f|_{S\cap U} = F|_{S\cap U}$ and $g|_{S\cap U} = G|_{S\cap U}$.

Thus, it is possible that the quasi-Poisson submanifold S of P has its own Poisson structure (that is given a priori) but it is not the one induced by the Poisson structure of P. For a discussion of these issues see [OrRa03], Sections 4.1.21–4.1.23.

Let $c \in C^\infty(S)$ be a Casimir function for the Poisson manifold $(S, \{\cdot,\cdot\}_S)$. Any extension $C \in C^\infty(P)$ of c will be called a *sub-Casimir* of $(C^\infty(P), \{\cdot,\cdot\})$.

Here is an example of the construction just described. Take some Casimir functions $C_1, \ldots, C_k \in C^\infty(P)$ of $(P, \{\cdot,\cdot\})$ and assume that a certain common level set S of these Casimirs is an embedded submanifold of P. It is easy to check that $B^\sharp(s)\left(T_s^*P\right) \subset T_sS$ for any $s \in S$ and hence S carries a unique Poisson bracket $(\{\cdot,\cdot\}_S)$ such that $(S, \{\cdot,\cdot\}_S)$ is a Poisson manifold with its own Casimir functions that extend to sub-Casimirs on P.

1.4. Leibniz systems

If in the definition of a Poisson manifold we drop the condition that the bracket $\{\cdot,\cdot\}$ induces a Lie algebra structure on $C^\infty(P)$ but we preserve the derivation property we obtain a *Leibniz manifold* [OrPl04]. The dynamical systems defined using Leibniz brackets include systems with dissipation, gradient systems, and nonholonomically constrained dynamical systems, among others. Let $(P,\{\cdot,\cdot\})$ be a Leibniz manifold and let h be a smooth function on P. There exist two vector fields X_h^R and X_h^L on P uniquely characterized by the relations

$$X_h^{\mathrm{R}}[f]=\{f,h\} \quad \text{and} \quad X_h^{\mathrm{L}}[f]=-\{h,f\}, \quad \text{for any } f\in C^\infty(P).$$

We will call X_h^{R} (respectively X_h^{L}) the *right* (respectively *left*) *Leibniz vector field* associated to the *Hamiltonian function* $h\in C^\infty(P)$. In this paper, the abbreviation X_h will always denote X_h^{R}. It should be noticed that if the Leibniz bracket $\{\cdot,\cdot\}$ is not skew-symmetric and $h\in C^\infty(P)$ is arbitrary then h is in general not a conserved quantity for X_h. Additionally, the characteristic distributions that one can define via $\{\cdot,\cdot\}$ using right and left Leibniz vector fields are in general not integrable and hence there is no analog of the symplectic stratification theorem for Leibniz manifolds. A function $f\in C^\infty(P)$ such that $\{f,g\}=0$ (respectively, $\{g,f\}=0$) for any $g\in C^\infty(P)$ is called a *left* (respectively, *right*) *Casimir* of the Leibniz manifold $(P,\{\cdot,\cdot\})$.

2. Stability in constrained and Poisson systems

In this section, we use some aspects of the geometry of Poisson and constrained systems to study the stability of their equilibria.

Let M be a manifold, $X\in\mathfrak{X}(M)$ a vector field, F_t the flow of X, and $m_{\mathrm{e}}\in M$ an equilibrium of X, that is, $X(m_{\mathrm{e}})=0$ or, equivalently, $F_t(m_{\mathrm{e}})=m_{\mathrm{e}}$ for all $t\in\mathbb{R}$. Recall that m_{e} is *stable*, or *Lyapunov stable*, if for any open neighborhood U of m_{e} in M there is an open neighborhood $V\subset U$ of m_{e} such that $F_t(m)\in U$ for any $m\in V$ and for any $t>0$. The equilibrium m_{e} is *asymptotically stable* if there is a neighborhood V of m_{e} such that $F_t(V)\subset F_s(V)$ whenever $t>s$ and $\lim_{t\to\infty} F_t(V)=m_{\mathrm{e}}$, that is, for any neighborhood W of m_{e} there is a $T>0$ such that $F_t(V)\subset W$ if $t\geqslant T$. If only the first condition holds and the inclusion is strict, that is, $F_t(V)\subsetneq F_s(V)$ whenever $t>s$, we say that m_{e} is *weakly asymptotically stable*. Note that

$$\text{asymptotic stability} \Rightarrow \text{weak asymptotic stability} \Rightarrow \text{Lyapunov stability}.$$

Asymptotic stability cannot occur in symplectic Hamiltonian systems due to Liouville's theorem; only Lyapunov stability is allowed. In the Poisson category, equilibria lying in zero dimensional symplectic leaves may be asymptotically stable. However, if the symplectic leaf that contains the equilibrium is at least two-dimensional, weak asymptotic stability is the most we can hope for.

The *linearization* of X at the equilibrium point m_{e} is the linear map $L : T_{m_{\mathrm{e}}}M \to T_{m_{\mathrm{e}}}M$ defined by $L(v) := \frac{d}{dt}\Big|_{t=0} \left(T_{m_{\mathrm{e}}}F_t(v)\right)$ where F_t is the flow of X and $v \in T_{m_{\mathrm{e}}}M$ is arbitrary. As is well known, the study of the spectrum of the linear map L gives relevant information about the stability of the equilibrium m_{e}. The equilibrium $m_{\mathrm{e}} \in M$ is *linearly stable* (respectively *unstable*) if the origin is a stable (respectively unstable) equilibrium for the linear dynamical system on $T_{m_{\mathrm{e}}}M$ defined by L. The equilibrium m_{e} is *spectrally stable* (respectively *unstable*) if the spectrum of the linear map L lies in the (strict) left-half plane or on the imaginary axis (respectively at least one eigenvalue has strictly positive real part). Lyapunov and linear stability imply spectral stability. If all the eigenvalues of L have strictly negative real part, that is, they lie in the (strict) left-half plane, the system is asymptotically stable.

2.1. Linearization of Poisson dynamical systems and linear stability

Consider a Hamiltonian vector field X_H on the Poisson manifold $(P, \{\cdot,\cdot\})$, let $z_{\mathrm{e}} \in P$ be an equilibrium of X_H, and $L : T_{z_{\mathrm{e}}}P \to T_{z_{\mathrm{e}}}P$ the linearization of X_H at z_{e}. If z_{e} is regular (in particular, when P is a symplectic manifold) there are restrictions on the eigenvalues of L that do not allow us to conclude the Lyapunov stability of z_{e} from its spectral stability (see, for instance, Theorem 3.1.17 in [AM78]). As will be shown below, this restriction disappears, in general, for equilibria lying on singular symplectic leaves.

In order to present the following lemma, whose proof is a straightforward computation, we recall that there exists a chart (U, φ) around any point $z \in P$ in the $2n + r$ dimensional Poisson manifold $(P, \{\cdot,\cdot\})$ such that $\varphi(z) = \mathbf{0}$ and that the associated local coordinates, denoted by $(q^1, \ldots, q^n, p_1, \ldots, p_n, z^1, \ldots, z^r)$, satisfy $\{q^i, q^j\} = \{p_i, p_j\} = \{q^i, z^k\} = \{p_i, z^k\} = 0$ and $\{q^i, p_j\} = \delta^i_j$, for all i, j, k such that $1 \leqslant i, j \leqslant n$, $1 \leqslant k \leqslant r$. For all such that k, l, $1 \leqslant k, l \leqslant r$, the Poisson bracket $\{z^k, z^l\}$ is a function of the local coordinates $z^1, \ldots, z^r$ exclusively and vanishes at z. Hence, the restriction of the bracket $\{\cdot,\cdot\}$ to the coordinates $z^1, \ldots, z^r$ induces a Poisson structure on an open neighborhood V of the origin in $\mathbb{R}^r$ whose Poisson tensor will be denoted by $\mathcal{R} \in \Lambda^2(V)$. This Poisson structure on V is called the *transverse Poisson structure* of $(P, \{\cdot,\cdot\})$ at z and is unique up to Poisson isomorphisms. The coordinates of the local chart that we just described are called *Darboux–Weinstein* coordinates [We83].

Lemma 2.1. *Let z_{e} be an equilibrium of the Hamiltonian dynamical system on the Poisson manifold $(P, \{\cdot,\cdot\})$ and let $(\mathbf{q}, \mathbf{p}, \mathbf{z})$ be a Darboux–Weinstein chart around z. Denote by $\mathbf{x} := (\mathbf{q}, \mathbf{p})$ and by J the $n \times n$ square matrix given by*

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

The linearization L of X_H at the equilibrium z_{e} in the coordinates $(\mathbf{x}, \mathbf{z})$ takes the form

$$L = \begin{pmatrix} \mathcal{S} & \mathcal{Q} \\ 0 & \mathcal{P} \end{pmatrix}, \tag{2.1}$$

where

$$\mathcal{S}^i_j = \sum_{p=1}^{2n} J^{ip} \frac{\partial^2 H}{\partial x^p \partial x^j}(\mathbf{0}, \mathbf{0}), \quad \mathcal{P}^k_l = \sum_{p=1}^{r} \frac{\partial \mathcal{R}^{kp}}{\partial z^l}(\mathbf{0}) \frac{\partial H}{\partial z^p}(\mathbf{0}, \mathbf{0}), \quad \text{and}$$

$$\mathcal{Q}^i_l = \sum_{p=1}^{2n} J^{ip} \frac{\partial^2 H}{\partial x^p \partial z^l}(\mathbf{0}, \mathbf{0}).$$

Proof. The result is obtained by differentiating the expression of the Hamiltonian vector field at the equilibrium in Darboux–Weinstein coordinates and by taking into account that the matrix J is constant, that $\mathcal{R}(\mathbf{0})$ is zero, and that $\mathcal{R}$ depends only on the $\mathbf{z}$ variables. □

We now use (2.1) to give a characterization of the structure of the eigenvalues of the linearized vector field L in the Poisson context. The proof of the following proposition is a straightforward computation.

Proposition 2.2. *In the situation described in the previous lemma denote by $\{\lambda_1, \ldots, \lambda_{2n}\}$ the eigenvalues of the infinitesimally symplectic matrix $\mathcal{S}$, counted with their multiplicities, and let $\{u_1, \ldots, u_{2n}\}$ be a basis of corresponding eigenvectors. Assume that the matrix $\mathcal{P}$ is diagonalizable, let $\{\mu_1, \ldots, \mu_r\}$ be its eigenvalues counted with their multiplicities, and $\{v_1, \ldots, v_r\}$ a basis of eigenvectors. Then the matrix L has eigenvalues $\{\lambda_1, \ldots, \lambda_{2n}, \mu_1, \ldots, \mu_r\}$. If for any eigenvalue μ_j we have that $(\mathcal{S} - \mu_j I)^{-1}\mathcal{Q}v_j$ is not empty then L is diagonalizable with corresponding basis of eigenvectors*

$$\{(u_1, 0), \ldots, (u_{2n}, 0), (-w_1, v_1), \ldots, (-w_r, v_r)\},$$

where $w_j \in (\mathcal{S} - \mu_j I)^{-1}\mathcal{Q}v_j$, $j = 1, \ldots, r$, are arbitrary but subjected to the condition that if $v_j = v_k$ then (w_j, v_j) and (w_k, v_k) are chosen to be linearly independent.

The eigenvalues $\{\lambda_1, \ldots, \lambda_{2n}\}$ satisfy the symplectic eigenvalue theorem since $\mathcal{S}$ is infinitesimally symplectic. However, the eigenvalues $\{\mu_1, \ldots, \mu_r\}$ may lie, in principle, anywhere in the complex plane. Hence Poisson dynamical systems may exhibit asymptotic behavior. There are three specific situations that should be singled out:

- None of the eigenvalues of $\mathcal{P}$ coincides with one of the eigenvalue of $\mathcal{S}$. In this case the matrices $(\mathcal{S} - \mu_j I)$, $1 \leqslant j \leqslant r$, are invertible and the whole linear system L is diagonalizable.
- $\mu_i = \lambda_j$ for some i, j but $(\mathcal{S} - \mu_i I)^{-1}\mathcal{Q}v_i$ is not empty. Then there is a passing of eigenvalues but they do not interact in the sense that they correspond to different blocks in the linearized system. We will call this situation *uncoupled passing*.
- If in the previous case $(\mathcal{S} - \mu_i I)^{-1}\mathcal{Q}v_i$ is empty then the linear system is not diagonalizable anymore and the passing of eigenvalues mixes blocks of the infinitesimally symplectic part and the transversal one. We will call this situation *coupled passing*.

With these remarks in mind, we get the following.

Proposition 2.3. *Let $(P, \{\cdot, \cdot\}, H)$ be a Poisson dynamical system and $z_{\mathrm{e}} \in P$ an equilibrium point of X_H. If the linearization L of X_H at z_{e} exhibits a coupled passing then the system is linearly unstable.*

Proof. The existence of a coupled passing implies the occurrence in L of a nondiagonal block in its Jordan canonical form. The flow of the linear dynamical system induced by L, when restricted to the space generated by the associated Jordan basis, exhibits an unstable behavior and the result follows. $\square$

Corollary 2.4. *Consider the linearization L of a Poisson dynamical system $(P, \{\cdot, \cdot\}, H)$ around an equilibrium $z_{\mathrm{e}} \in P$ lying on a regular symplectic leaf $\mathcal{L}$. Let $\{\lambda_1, \ldots, \lambda_{2n}\}$ be the eigenvalues of the infinitesimally symplectic block $\mathcal{S}$. Then*

(i) $\mathcal{P} = 0$.
(ii) *The vectors $u \in T_{z_{\mathrm{e}}}P$ that satisfy $Lu = \lambda u$ for some $\lambda \neq 0$ lie in $T_{z_{\mathrm{e}}}\mathcal{L}$. In particular, the unstable directions of L are tangent to the symplectic leaf of P that contains the equilibrium.*
(iii) *If $S^{-1}\mathcal{Q}v_j$ is not empty for any v_j as in Proposition 2.2 then 0 is the only eigenvalue in addition to $\{\lambda_1, \ldots, \lambda_{2n}\}$.*

Proof. The first part follows from the expression for $\mathcal{P}$ provided in Lemma 2.1 and from the fact that $\mathcal{R} = 0$ in an open neighborhood of z_{e} that contains only regular points. The unstable directions are the vectors in the eigenspaces corresponding to strictly positive eigenvalues. Then the points (ii) and (iii) follow from the expression of L in Lemma 2.1 using that on the set of regular points $\mathcal{R} = \mathcal{P} = 0$. $\square$

2.2. Nonlinear stability in constrained and Poisson dynamical systems

As noted in the previous subsection, the array of linear tools available to conclude nonlinear stability of equilibria of a Poisson dynamical system is very limited. In this section we will formulate a result for constrained systems that, in the Poisson case, provides a sufficient condition for such equilibria to be Lyapunov or weakly asymptotically stable. This result is inspired by the use of first integrals of motion in Hamiltonian systems and is related to the classical energetics methods (also called Dirichlet criteria) in [A66,Paal04]. Our approach builds on an improvement of the classical result in [A66] that was carried out in [Or98] (see Corollary 4.11 in [OrRa99b]).

The proof of our main result will be based on a classical result of Lyapunov that states that *if* $m_{\rm e} \in M$ *is an equilibrium of the vector field* $X \in \mathfrak{X}(M)$ *with flow* F_t *and there exists a positive function* $L \in C^\infty(U)$ *around* $m_{\rm e}$, *with* U *an open neighborhood of* $m_{\rm e}$, *such that* $\dot{L}(m) := \frac{d}{dt}\big|_{t=0} L(F_t(m)) \leqslant 0$, *for any* $m \in U \setminus \{m_{\rm e}\}$, *then* $m_{\rm e}$ *is a Lyapunov stable equilibrium*. We recall that a function $f \in C^\infty(M)$ is said to be *positive* around $m_{\rm e} \in M$ if $f(m_{\rm e}) = 0$ and there is an open neighborhood $U_{m_{\rm e}}$ of $m_{\rm e}$ such that $f(m) > 0$, for all $m \in U_{m_{\rm e}} \setminus \{m_{\rm e}\}$. *If* $\dot{L}(m) < 0$ *for all* $m \in U_{m_{\rm e}} \setminus \{m_{\rm e}\}$, *then* $m_{\rm e}$ *is asymptotically stable*. See e.g. Theorem 1, Chapter 9, Section 3 in [HS74] for a proof of these statements; the infinite dimensional versions of these assertions can be found in Theorems 4.3.11 and 4.3.12 of [AMR88]. Any positive function L in the statement of Lyapunov's theorem is usually called a *Lyapunov function*. Its construction for specific dynamical systems is by itself a very active research subject.

In the case of Hamiltonian mechanics, the Hamiltonian and the Casimirs of the Poisson phase space are natural candidates to be used in Lyapunov's theorem. If, additionally, the system has a symmetry to which one can associate a momentum map, its components are conserved quantities that sometimes can be used for the same purpose. The use of all conserved quantities of a dynamical system in the study of the stability of equilibria to form Lyapunov functions is known under the name of energy–momentum methods. However, it should be noted that, apart from conserved quantities, Lyapunov's theorem can be applied with the more general class of functions whose time derivative is strictly negative. The existence of these functions implies the asymptotic stability of the equilibrium in question. In the symplectic context this is impossible. This behavior, allowed for Poisson Hamiltonian systems, is used in the main theorem of this subsection and illustrated in some of the examples that follow.

In the sequel we will use the following notation. Let P be a smooth manifold, $f \in C^\infty(P)$ a smooth function, $z_{\rm e} \in P$ a critical point of f (that is, $\mathbf{d}f(z_{\rm e}) = 0$), and U an open neighborhood of $z_{\rm e}$. The *Hessian* of f at the critical point $z_{\rm e}$ is the symmetric bilinear form $\mathbf{d}^2 f(z_{\rm e}) : T_{z_{\rm e}}P \times T_{z_{\rm e}}P \to \mathbb{R}$ given by $\mathbf{d}^2 f(z_{\rm e})(v, w) := v[W[f]]$, where $v, w \in T_{z_{\rm e}}P$ and $W \in \mathfrak{X}(U)$ is an arbitrary extension of w to a vector field on U. The fact that $z_{\rm e}$ is a critical point of f ensures that this definition is independent of the extension W of w. Additionally, given a vector field $X \in \mathfrak{X}(P)$ with flow F_t we define $\dot{f}(z) := X[f](z) = \frac{d}{dt} f(F_t(z))$, for any $f \in C^\infty(P)$ and $z \in P$.

Theorem 2.5. *Let* $X \in \mathfrak{X}(P)$ *be a vector field on the manifold P. Let* $z_{\rm e}$ *be an equilibrium point of X and* $C_0, C_1, \ldots, C_k : P \to \mathbb{R}$ *conserved quantities of X, that is* $X[C_i] = 0$, $i \in \{0, \ldots, k\}$. *Let* $F : P \to \mathbb{R}$ *be a function such that* $F(z_{\rm e}) = 0$ *and that satisfies the conditions*:

(i) $X[F^2] \leqslant 0$,
(ii) $X[F](y) \leqslant 0$ *for all the points* $y \in P \setminus \{z_{\rm e}\}$ *satisfying* $X[F^2](y) = 0$.

Assume that there exist constants $\{\lambda_0, \lambda_1, \ldots, \lambda_k, \mu\}$ *such that*

$$\mathbf{d}(\lambda_0 C_0 + \lambda_1 C_1 + \cdots + \lambda_k C_k + \mu F)(z_e) = 0$$

and the quadratic form

$$\mathbf{d}^2(\lambda_0 C_0 + \lambda_1 C_1 + \cdots + \lambda_k C_k + \mu F)|_{W \times W}(z_e) \tag{2.2}$$

is positive definite with

$$W := \ker \mathbf{d}C_0(z_e) \cap \ker \mathbf{d}C_1(z_e) \cap \cdots \cap \ker \mathbf{d}C_k(z_e).$$

Then z_e *is a weakly asymptotically stable equilibrium* (*and hence Lyapunov stable*). *If the inequality* $X[F^2](z) \leqslant 0$ *is strict for every* $z \in P \setminus \{z_e\}$ *then* z_e *is asymptotically stable*.

Proof. Consider the functions $l_1, l_2 \in C^\infty(P)$ defined by

$$l_1(z) := \sum_{j=0}^{k} \left(\lambda_j C_j(z) + \mu F(z)\right) - \left(\lambda_j C_j(z_e)\right),$$

$$l_2(z) := \sum_{j=0}^{k} \frac{1}{2} \left((C_j(z) - C_j(z_e))^2 + F(z)^2\right).$$

Notice that $l_1(z_e) = 0$ and that, by hypothesis, $\mathbf{d}l_1(z_e) = 0$ which implies that $\mathbf{d}^2 l_1(z_e)$ is well defined. Moreover, hypothesis (2.5) is equivalent to $\mathbf{d}^2 l_1(z_e)|_{W\times W}$ being positive definite. Additionally, $l_2(z_e) = 0$, $\mathbf{d}l_2(z_e) = 0$, and hence $\mathbf{d}^2 l_2(z_e)$ is well defined. A straightforward computation shows that $\mathbf{d}^2 l_2(z_e)$ is positive semidefinite with kernel equal to the space W. A result due to Patrick (see [Pa92]) shows that in these circumstances there exists a constant $r > 0$ such that for any $\varepsilon \in (0, r]$ the Hessian $\mathbf{d}^2(\varepsilon l_1 + l_2)(z_e)$ is positive definite.

Let $L_\varepsilon := \varepsilon l_1 + l_2$. The positive definiteness of $\mathbf{d}^2 L_\varepsilon(z_e)$ implies that L_ε is a positive function on an open neighborhood U of z_e whose level sets are, by the Morse lemma, diffeomorphic to concentric spheres centered at the equilibrium z_e. Additionally, conditions (i) and (ii) imply that the constant ε can be chosen small enough so that the time derivative

$$\dot{L}_\varepsilon(z) = \tfrac{1}{2} X[F^2](z) + \varepsilon\mu X[F](z) \leqslant 0 \tag{2.3}$$

for any $z \in P$. This implies that if F_t is the flow of X_H, the basis of open neighborhoods of z_e given by the sets $U_\lambda := L_\varepsilon^{-1}([0, \lambda))$, with λ small enough, satisfies $F_t(U_\lambda) \subseteq F_s(U_\lambda)$, provided that $t \geqslant s$. This proves the weak asymptotic stability of z_e.

If $X[F^2](z) < 0$ for every $z \in P \setminus \{z_e\}$ then ε can be chosen so that the positive function L_ε is such that $\dot{L}_\varepsilon(z) < 0$ for any $z \in P \setminus \{z_e\}$ (see (2.3)). Lyapunov's theorem proves the asymptotic stability of z_e. $\square$

Remark 2.6. The most efficient way to apply Theorem 2.5 in order to establish the stability of a given equilibrium consists of looking at the system obtained by restriction of the original one to an arbitrarily small neighborhood of the equilibrium. The advantages of proceeding in this way are based on the fact that the restricted system has, in general, more conserved quantities than the original one. We illustrate this remark with the following specific example.

Consider the manifold $P := \mathbb{T}^2 \times \mathbb{R}$ endowed with the Poisson structure given by the tensor that in coordinates (θ, φ, x) is expressed as

$$B(\theta, \varphi, x) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -\alpha \\ -1 & \alpha & 0 \end{pmatrix}, \qquad \alpha \in \mathbb{R} \setminus \mathbb{Q}.$$

Let $H \in C^\infty(P)$ be the function defined by $H(\theta, \varphi, x) := x^2 - \cos\theta$. The associated Hamiltonian vector field $X_H = 2x\frac{\partial}{\partial\theta} - 2\alpha x\frac{\partial}{\partial\varphi} - \sin\theta\frac{\partial}{\partial x}$ has an equilibrium at the point $z_e := (0, 0, 0)$ whose stability we show using Theorem 2.5. Even though the Poisson manifold P has no globally defined Casimir functions, any locally defined function of the form $C = \alpha\theta + \varphi$ is a local Casimir. We can use this local Casimir to establish the Lyapunov stability of z_e. Indeed, $\mathbf{d}H(z_e) = 0$ and $\mathbf{d}^2H(z_e)|_{W\times W} > 0$, with $W = \ker \mathbf{d}C(z_e)$. In Section 3.2, we will describe a mechanical system that is closely related to this example.

Example 2.7 (*Double bracket dissipation*). Morrison [Mo86] and Brockett [Br88,Br93] have proposed the modelling of certain dissipative phenomena by adding a symmetric bracket to a known skew-symmetric one, that is,

$$\{\cdot, \cdot\}_{\text{Leibniz}} = \{\cdot, \cdot\}_{\text{skew}} + \{\cdot, \cdot\}_{\text{sym}},$$

where the bracket $\{\cdot, \cdot\}_{\text{skew}}$ is skew-symmetric, $\{\cdot, \cdot\}_{\text{sym}}$ is symmetric, and hence the sum is a Leibniz bracket. This scheme allows the modeling of a surprising number of physical examples. The reader is encouraged to check with [Mars92,Blal96a] for an account of applications and references in this direction.

An example that fits into this framework is the equation arising from the Landau–Lifschitz model for the magnetization vector $\mathbf{M}$ in an external vector field $\mathbf{B}$,

$$\dot{\mathbf{M}} = \gamma\mathbf{M} \times \mathbf{B} + \frac{\beta}{\|\mathbf{M}\|^2}(\mathbf{M} \times (\mathbf{M} \times \mathbf{B})), \tag{2.4}$$

where γ and β are physical parameters. This equation is Leibniz in our sense if we take the Leibniz bracket on $\mathbb{R}^3$ given by the sum of the two brackets

$$\{f, g\}_{\text{skew}}(\mathbf{M}) := \mathbf{M} \cdot (\nabla f(\mathbf{M}) \times \nabla g(\mathbf{M})) \quad \text{and}$$
$$\{f, g\}_{\text{sym}}(\mathbf{M}) := \frac{\beta(\mathbf{M} \times \nabla f(\mathbf{M}))(\mathbf{M} \times \nabla g(\mathbf{M}))}{\gamma \|\mathbf{M}\|^2},$$

where the symbol $\times$ denotes the standard cross product on $\mathbb{R}^3$ and ∇ is the Euclidean gradient. With this bracket the differential equation (2.4) corresponds to the expression of the Leibniz vector field determined by the function

$$H(\mathbf{M}) = \gamma \mathbf{B} \cdot \mathbf{M}.$$

Assume that $\mathbf{B}$ is constant and of the form $\mathbf{B} = (0, 0, 1)$. The system has then an equilibrium at the point $m_0 = (0, 0, M_0)$ for every $M_0 \in \mathbb{R}$. We will assume that M_0 is different from zero so that there are no singularities in the definition of the bracket. If we compute the linearization of X_H at the equilibrium we obtain

$$L = \begin{pmatrix} \frac{-\beta}{M_0} & -\gamma & 0 \\ \gamma & \frac{-\beta}{M_0} & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

whose eigenvalues are

$$\mu_1 = \frac{-\beta}{M_0} + i\gamma, \qquad \mu_2 = \frac{-\beta}{M_0} - i\gamma, \qquad \text{and} \qquad \mu_3 = 0.$$

If $\beta/M_0 < 0$ then the equilibrium m_0 is unstable since there are eigenvalues with positive real parts. If $\beta/M_0 > 0$ the eigenvalues with negative real part correspond to the subspace generated by the vectors $(1, 0, 0)$ and $(0, 1, 0)$. This suggests the choice $F(\mathbf{M}) = \frac{1}{2}(M_1^2 + M_2^2)$ to be used as the function F in Theorem 2.5. It is easy to check that if $\beta/M_0 > 0$ then there exists an open neighborhood of m_0 on which F and F^2 satisfy conditions (i) and (ii) in the statement of Theorem 2.5. This follows from the equalities

$$\dot{F} = \{F, H\} = -\frac{(M_1^2 + M_2^2)M_3\beta}{||\mathbf{M}||^2} \quad \text{and} \quad \{F^2, H\} = -2\frac{(M_1^2 + M_2^2)^2 M_3\beta}{||\mathbf{M}||^2}.$$

The system has a conserved quantity given by

$$C(\mathbf{M}) = ||\mathbf{M}||^2,$$

which is in fact a left Casimir for the Leibniz structure. The equality

$$\mathbf{d}(\lambda_0 C + \mu F)(m_0) = 0$$

is satisfied if and only if $\lambda_0 = 0$. Take $\mu = 1$. Then $W = \ker \mathbf{d}C(m_0) = \operatorname{span}\{(1,0,0), (0,1,0)\}$ and $\mathbf{d}^2 F(m_0)\big|_{W\times W} > 0$. *The equilibrium* $m_0 = (0,0,M_0)$ *is thus weakly asymptotically stable whenever* β/M_0 *is positive*.

Notice that we did not use the Hamiltonian since it is not a conserved quantity for this system. Notice also that even though λ_0 must vanish in order for the critical point condition to be satisfied, the conserved quantity C contributes in an essential way by making the subspace W sufficiently small for the condition (2.5) to hold. Had we ignored C in the construction of W the quadratic form $\mathbf{d}^2 F(m_0)\big|_{W\times W}$ would be only positive semidefinite and hence the theorem would not apply.

In the following corollary we reformulate Theorem 2.5 for Poisson manifolds.

Corollary 2.8. *Let* $(P, \{\cdot,\cdot\}, H)$ *be a Poisson dynamical system. Let* z_e *be an equilibrium point of* X_H *and* $C_1, \ldots, C_k : P \to \mathbb{R}$ *conserved quantities of* X_H, *that is* $\{C_i, H\} = 0$, $i \in \{1, \ldots, k\}$. *Let* $F : P \to \mathbb{R}$ *be a function such that* $F(z_e) = 0$ *and that satisfies the conditions*:

(i) $\{F^2, H\} \leqslant 0$,
(ii) $\{F, H\}(y) \leqslant 0$ *for all the points* $y \in P \setminus \{z_e\}$ *satisfying* $\{F^2, H\}(y) = 0$.

Assume that there exist constants $\{\lambda_0, \lambda_1, \ldots, \lambda_k, \mu\}$ *such that*

$$\mathbf{d}(\lambda_0 H + \lambda_1 C_1 + \cdots + \lambda_k C_k + \mu F)(z_e) = 0$$

and the quadratic form

$$\mathbf{d}^2(\lambda_0 H + \lambda_1 C_1 + \cdots + \lambda_k C_k + \mu F)|_{W\times W}(z_e) \tag{2.5}$$

is positive definite with

$$W := \ker \mathbf{d}H(z_e) \cap \ker \mathbf{d}C_1(z_e) \cap \cdots \cap \ker \mathbf{d}C_k(z_e).$$

Then z_e *is a weakly asymptotically stable equilibrium* (*and hence Lyapunov stable*). *If the inequality* $\{F^2, H\} \leqslant 0$ *is strict for every* $z \in P\setminus\{z_e\}$ *then* z_e *is asymptotically stable* (*this can only happen if the symplectic leaf that contains the equilibrium is trivial*).

Remark 2.9. The main differences between this result (Corollary 2.8) and those already existing in the literature are:

(i) It takes advantage of the possible existence of strict Lyapunov functions and hence is capable of obtaining the Lyapunov stability of an equilibrium as a corollary of an asymptotically stable behavior. This feature allows us to prove stability in some examples where no other available energy method is applicable.

In order to illustrate this point consider the following example. The two dimensional Toda lattice admits a Poisson formulation [Bl00] by taking the bracket $\{x, y\} = -x$ and the Hamiltonian function $H(x, y) = x^2 + y^2$. The equations of the system are $\dot{x} = -2xy$ and $\dot{y} = 2x^2$. This system has an equilibrium point at $z_e = (0, b)$ for any $b \in \mathbb{R}$. The equilibrium $(0, 0)$ is obviously Lyapunov stable since $\mathbf{d}H(0, 0) = 0$ and $\mathbf{d}^2H(0, 0) > 0$. The equilibria of the form $z_e = (0, b)$ with $b > 0$ are weakly asymptotically stable. This can be proved using the previous theorem by taking the Hamiltonian as conserved quantity and the function $F(x, y) := x$. The function F satisfies hypotheses (i) and (ii) in Theorem 2.5 since $\{F^2, H\} = -4x^2y \leqslant 0$ and $\{F, H\} = -2xy = 0$ when $\{F^2, H\} = 0$, in an open neighborhood of $z_e = (0, b)$ with $b > 0$. If $b < 0$ the equilibrium is unstable since the linearization has an eigenvalue with positive real part. We emphasize that the stability of the points in the case $b > 0$ are uniquely due to their weak asymptotically stable behavior.

(ii) Unlike the approach taken in the treatment of many standard examples (see for instance [MaRa99]) this theorem shows that one does not need to take arbitrary functions of the conserved quantities in the expression (2.5). Indeed, only linear combinations are needed. This is a consequence of the fact that the form whose definiteness needs to be studied is restricted to the space W.

(iii) Since the constants $\{\lambda_0, \lambda_1, \ldots, \lambda_k, \mu\}$ are allowed to be zero we have the freedom not to use a local conserved quantity in the definiteness condition (2.5) but to still take advantage of its existence to shrink the space W. This is an improvement with respect to the results in [Or98] (see Corollary 4.11 in [OrRa99b]).

In order to visualize this better consider the following example. Let $(\mathbb{R}^3, \{\cdot, \cdot\}, H)$ be the Poisson dynamical system whose Poisson bracket is given by the Poisson tensor that in Euclidean coordinates takes the form

$$B(x, y, z) = \begin{pmatrix} 0 & 0 & y \\ 0 & 0 & -x \\ -y & x & 0 \end{pmatrix}$$

and where $H(x, y, z) = az$, with $a \in \mathbb{R}$ a nonzero constant. The equations of motion are

$$\dot{x} = ay, \qquad \dot{y} = -ax, \qquad \text{and} \qquad \dot{z} = 0.$$

The function $C(x, y, z) = \frac{1}{2}\left(x^2 + y^2\right)$ is a Casimir for this Poisson structure and every point of the form $(0, 0, z_0)$ is an equilibrium of X_H. Note that $\mathbf{d}(H - \lambda C)(0, 0, z_0) \neq 0$ for any $\lambda \in \mathbb{R}$. Nevertheless, we can still apply the previous theorem to conclude the Lyapunov stability of $(0, 0, z_0)$ by taking the combination $\lambda_0 H + \lambda_1 C$ with $\lambda_0 = 0$ and $\lambda_1 = 1$. With these choices, $W = \ker(\mathbf{d}H(0, 0, z_0))$

and $\mathbf{d}^2 C(0,0,z_0)|_{W\times W}$ is positive definite. The stability of these equilibria can also be handled using the topological methods in [Paal04].

Remark 2.10. In most Hamiltonian applications, the conserved quantities in the statement of the theorem are local Casimir functions, components of momentum maps, and the Hamiltonian. A good way to find the functions F is to look for purely negative eigenvalues of the linearization of X_H at the equilibrium $z_{\rm e}$ that do not have a positive counterpart, as will be shown below. Notice that by Corollary 2.4 this is only possible when the equilibrium $z_{\rm e}$ is lying on a singular symplectic leaf of the Poisson manifold. More explicitly, suppose that the linearization has such a negative eigenvalue $-\lambda$ with eigenvector v. Take local coordinates $(y_1,\ldots,y_n)$ such that $v=\frac{\partial}{\partial y_n}$. Since the function $F_v(y_1,\ldots,y_n):=y_n$ satisfies $\{y_n^2,H\}=X_H[y_n^2]=2y_n\dot{y}_n=-2\lambda y_n^2+\text{h.o.t.}$, it is a good candidate to be used as the function F in the statement of the theorem. This procedure has been used in the first example in Remark 2.9.

2.3. Ideal reduction and ideal stability for Poisson systems

We start this section by describing new Poisson structures on some submanifolds of a Poisson manifold that can be obtained by looking at the ideals of its Poisson algebra of smooth functions. We will refer to the construction that will be presented as *ideal reduction* for it is a particular case of the Poisson reduction procedures in [MaRa86,OrRa98,OrRa03].

This reduction technique is used later in this section to define a weaker notion of stability, called $\mathcal{I}$-*stability*, and to establish a sufficient condition for it to hold. As the examples in the next section show, the use of $\mathcal{I}$-stability is a very sensible way to deal with the physically relevant stability properties of equilibria in Hamiltonian systems subjected to semiholonomic constraints.

Let P be a smooth manifold and $\mathcal{F}\subset C^\infty(P)$ be a family of smooth functions. Denote by $\mathcal{V}_{\mathcal{F}}\subset P$ the *vanishing subset* of $\mathcal{F}$, defined as the intersection of the zero level sets of all the elements of $\mathcal{F}$. For a subset $S\subset P$ define its *vanishing ideal* $\mathcal{I}(S)$ as the set of functions $f\in C^\infty(P)$ such that $f(S)=\{0\}$. Notice that $\mathcal{I}(S)$ is obviously an ideal of $C^\infty(P)$ with respect to the standard multiplication of functions. Notice also that for every subset $S\subset P$ and for every ideal $\mathcal{J}\in C^\infty(P)$ we have $S\subset\mathcal{V}_{\mathcal{I}(S)}$ and $\mathcal{J}\subset\mathcal{I}\left(\mathcal{V}_{\mathcal{J}}\right)$. These inclusions are in general strict. However, if S is a closed embedded submanifold of P then the first inclusion is actually an equality due to the smooth version of Urysohn's lemma. Moreover, in this particular case, the quotient algebra $C^\infty(P)/\mathcal{I}(S)$ can be identified with $C^\infty(S)$, the algebra of smooth functions on S with respect to its own smooth manifold structure, via the map that assigns to any $f\in C^\infty(S)$ the element $\pi(F)\in C^\infty(P)/\mathcal{I}(S)$, where $F\in C^\infty(P)$ is an arbitrary extension of f and $\pi: C^\infty(P)\to C^\infty(P)/\mathcal{I}(S)$ is the projection. We will say that an ideal $\mathcal{I}\subset C^\infty(P)$ is *regular* if its vanishing set $\mathcal{V}_{\mathcal{I}}\subset P$ is a closed and embedded submanifold of P.

In the sequel we will focus our attention on finitely generated Poisson ideals. Let $(P,\{\cdot,\cdot\})$ be a Poisson manifold and $\mathcal{F}=\{f_1,\ldots,f_n\}\subset C^\infty(P)$ be a finite family of elements in $C^\infty(P)$. We will say that $\mathcal{F}$ generates a *Poisson ideal* if for any function $f \in C^\infty(P)$ and any $i \in \{1, \ldots, n\}$ there exist functions $\{h_{i1}, \ldots, h_{in}\} \subset C^\infty(P)$ such that

$$\{f, f_i\} = \sum_{j=1}^{n} h_{ij} f_j.$$

Denoting

$$\mathcal{I}(\mathcal{F}) := \left\{ \sum_{k=1}^{n} g_k f_k \,\middle|\, g_k \in C^\infty(P) \right\}$$

note that the condition above is equivalent to the statement that $\mathcal{I}(\mathcal{F})$ is an ideal in the Poisson algebra $C^\infty(P)$, that is, it is an ideal relative to both the usual multiplication of functions as well as the Lie bracket $\{\cdot, \cdot\}$. Note that *if the vanishing subset $\mathcal{V}_{\mathcal{F}}$ of $\mathcal{F}$ is an embedded submanifold of P then $\mathcal{V}_{\mathcal{F}}$ is a quasi-Poisson submanifold of P*. Indeed, for any $f \in C^\infty(P)$, $f_i \in \mathcal{F}$, and $z \in \mathcal{V}_{\mathcal{F}}$, there exist functions $\{h_1, \ldots, h_n\} \subset C^\infty(P)$ such that

$$\langle \mathbf{d} f_i(z), X_f(z) \rangle = \{f_i, f\}(z) = \sum_{j=1}^{n} h_j(z) f_j(z) = 0,$$

which shows that $B^\sharp(z)(T_z^* P) \subset T_z \mathcal{V}_F$, as required. Since the embedded submanifold $\mathcal{V}_{\mathcal{F}}$ is quasi-Poisson, it has a Poisson bracket $\{\cdot, \cdot\}_{\mathcal{V}_F}$ given by $\{f, g\}_{\mathcal{V}_F}(z) := \{F, G\}(z)$, where $F, G \in C^\infty(P)$ are arbitrary local extensions of $f, g \in C^\infty(\mathcal{V}_F)$ around the point $z \in \mathcal{V}_F$. We recall that the extensions to P of the Casimir functions of $(\mathcal{V}_F, \{\cdot, \cdot\}_{\mathcal{V}_F})$ are called sub-Casimirs of $(P, \{\cdot, \cdot\})$.

The construction that we just carried out can be locally reversed, that is, given an injectively immersed quasi-Poisson submanifold S of $(P, \{\cdot, \cdot\})$ any point $z \in S$ has an open neighborhood V_z of z in S such that the vanishing ideal $\mathcal{I}(V_z)$ is a Poisson ideal generated by a finite family of smooth functions on P with $\operatorname{codim} S$ elements. Indeed, choose V_z small enough so that it is an embedded submanifold of P and that, at the same time, is contained in the domain of a submanifold chart (U_z, φ) of P. With this choice we can write $U_z \simeq W_1 \times W_2$ and $V_z \simeq W_1 \times \{0\}$, where W_1 and W_2 are open neighborhoods of the origin in two finite dimensional vector spaces of dimensions $\dim S$ and $\operatorname{codim} S$, respectively. If we denote the elements of W_2 by $(x_1, \ldots, x_{\operatorname{codim} S})$ then any arbitrary extensions $F_1, \ldots, F_{\operatorname{codim} S} \in C^\infty(P)$ of the coordinate functions $f_1 = x_1, \ldots, f_{\operatorname{codim} S} = x_{\operatorname{codim} S}$ to the manifold P generate $\mathcal{I}(V_z)$ and form a Poisson ideal. Indeed, since V_z is an embedded quasi-Poisson submanifold of P, we have for any $F \in C^\infty(P)$ and any $z' \in V_z$

$$\{F_i, F\}(z') = \{f_i, F|_{V_z}\}_{V_z}(z') = 0$$

since $f_i|_{V_z} \equiv 0$.

Some of the ideas that we just introduced play a very important role in the algebraic approach to Poisson geometry. The reader interested in these kind of questions is encouraged to check with [Va96] and references therein.

Definition 2.11. Let $(P, \{\cdot, \cdot\}, H)$ be a Poisson dynamical system and let $\mathcal{I}$ be a regular Poisson ideal, that is, the vanishing set $\mathcal{V}_{\mathcal{I}}$ is a closed and embedded submanifold of P. Consider the reduced Poisson system $(\mathcal{V}_{\mathcal{I}}, \{\cdot, \cdot\}_{\mathcal{V}_{\mathcal{I}}}, h)$ where $h \in C^\infty(\mathcal{V}_{\mathcal{I}})$ is defined by $h := H \circ i$ with $i : \mathcal{V}_{\mathcal{I}} \hookrightarrow P$ the inclusion. Assume that $z_e \in \mathcal{V}_{\mathcal{I}}$ is an equilibrium point for the Poisson dynamical system $(P, \{\cdot, \cdot\}, H)$ and hence also for $(\mathcal{V}_{\mathcal{I}}, \{\cdot, \cdot\}_{\mathcal{V}_{\mathcal{I}}}, h)$. We say that $z_e \in \mathcal{V}_{\mathcal{I}} \subset P$ is an *$\mathcal{I}$-stable equilibrium* if any of the two following equivalent conditions hold:

(i) z_e is a stable equilibrium for the reduced Poisson dynamical system $(\mathcal{V}_{\mathcal{I}}, \{\cdot, \cdot\}_{\mathcal{V}_{\mathcal{I}}}, h)$;
(ii) for any open neighborhood U of z_e in P, there is an open neighborhood V of z_e in P such that if F_t is the flow of X_H, then $F_t(z) \in U \cap \mathcal{V}_{\mathcal{I}}$ for any $z \in V \cap \mathcal{V}_{\mathcal{I}}$ and for any $t > 0$.

The equilibrium z_e is *$\mathcal{I}$-unstable* if z_e is an unstable equilibrium for the reduced Poisson dynamical system $(\mathcal{V}_{\mathcal{I}}, \{\cdot, \cdot\}_{\mathcal{V}_{\mathcal{I}}}, h)$. It is obvious that $\mathcal{I}$-instability implies Lyapunov instability on the whole space.

Theorem 2.12. *Let $(P, \{\cdot, \cdot\}, H)$ be a Poisson dynamical system with an equilibrium at the point $z_e \in P$ and let $U \subset P$ be an open neighborhood around z_e. Assume that there exists a regular Poisson ideal $\mathcal{I}$ generated by the functions $G_1, \dots, G_m \in C^\infty(P)$ with sub-Casimirs $F_1, \dots, F_r \in C^\infty(P)$ such that $z_e \in \mathcal{V}_{\mathcal{I}}$. Suppose that the functions $C_0 := H, C_1, \dots, C_n \in C^\infty(P)$ are conserved by the flow of X_H and that, additionally, there exist constants $\lambda_1, \dots, \lambda_n, \mu_1, \dots, \mu_r, \nu_1, \dots, \nu_m$ such that*

(i) *$H_1 := \sum_{i=0}^n \lambda_i C_i + \sum_{j=1}^r \mu_j F_j + \sum_{k=1}^m \nu_k G_k$ has a critical point at z_e, and*
(ii) *the Hessian of H_1 at z_e is positive definite when restricted to the subspace W defined by*

$$W = \bigcap_{i=0}^{n} \ker(\mathbf{d}C_i(z_e)) \bigcap_{j=1}^{r} \ker(\mathbf{d}F_j(z_e)) \bigcap_{k=1}^{m} \ker(\mathbf{d}G_k(z_e)).$$

Then z_e is an $\mathcal{I}$-stable equilibrium.

Proof. The hypotheses in the statement of the theorem imply that the equilibrium z_e of the reduced system $(\mathcal{V}_{\mathcal{I}}, \{\cdot, \cdot\}_{\mathcal{V}_{\mathcal{I}}}, H \circ i)$, with $i : \mathcal{V}_{\mathcal{I}} \hookrightarrow P$ the inclusion, satisfies the hypotheses of Theorem 2.5 and hence is Lyapunov stable on $\mathcal{V}_{\mathcal{I}}$, which implies that z_e is $\mathcal{I}$-stable. □

3. Examples

3.1. A light Chaplygin sleigh on a cylinder

The following example was formulated in [Mar95] in the context of nonholonomically constrained systems. In that work the author found an equilibrium that exhibits asymptotically stable behavior. We will study the stability of all the equilibria of this system as well as of its relative equilibria with respect to a circle symmetry of the system that will be introduced later on. We will apply the Lyapunov stability methods presented in the previous sections. This example is based on a real mechanical system that illustrates the theory particularly well since it exhibits equilibria that are not critical points of the Hamiltonian or of any other conserved function and, nevertheless, Theorem 2.5 still allows us to establish the Lyapunov stability of some dynamical elements and, in some cases, asymptotic stability. There is also an equilibrium to which none of the stability methods in the paper apply but that, after ideal reduction, is shown to be unstable and hence unstable in the whole space.

We will start the presentation by explicitly carrying out in this particular example Marle's reduction procedure for nonholonomically constrained systems. The reader is encouraged to check with the original references [Mar95,Mar98] in order to find various technical details that we will omit here.

3.1.1. Description of the system

The configuration space is given by the points (x, θ) on a cylinder $Q := \mathbb{R} \times S^1$. The Lagrangian of the system is just the kinetic energy $L = \frac{1}{2}(\dot{x}^2 + \dot{\theta}^2) \in C^\infty(TQ)$. The system is constrained to move subject to the semiholonomic constraint $\dot{x} + x\dot{\theta} = 0$. The term "semiholonomic" means that the distribution that describes the constraint is integrable with integral leaves that are not necessarily embedded submanifolds.

This system approximates a simple mechanical system in a certain regime that can be physically realized in the following way. Take a Chaplygin sleigh moving in the interior of a cylinder (we are assuming that all the physical constants of the system are equal to 1). The configuration space of this system consists of the points $(x, \theta, \varphi) \in Q' := \mathbb{R} \times S^1 \times S^1$, where the coordinates (x, θ) on the cylinder indicate the position of the Chaplygin sleigh. The dynamics of this system is determined by the Lagrangian L' on TQ' given by $L' = \frac{1}{2}(\dot{x}^2 + \dot{\theta}^2 + I_\varphi \dot{\varphi}^2)$, where I_φ is the moment of inertia of the sleigh, together with the nonholonomic constraint $\dot{x} \cos \varphi - \dot{\theta} \sin \varphi = 0$. Assume now that we add a new holonomic constraint $\tan \varphi = x$. Notice that even if the first constraint was not integrable, the superposition of the two constraints *is* integrable. In this case the dynamics can be described by restricting the system to a new configuration space $\bar{Q} \subset Q'$ which is actually an integral manifold of the distribution that describes the holonomic constraint. Moreover, it is easy to see that we can restrict the system to the integral manifold of *any* subset of integrable constraints, obtaining a new holonomically constrained system. In this case, we restrict the system described by the Lagrangian L' on TQ' to the integral submanifold $Q \subset Q'$ by using the holonomic constraint $\tan \varphi = x$. Assuming $I_\varphi \ll 1$ and restricting our study to points such that $x \ll 1$, the example that we will be presenting is a good approximation of this mechanical system. Marle [Mar95] considers the same mechanical realization of these equations but he sets $I_\varphi = 0$ from the beginning of his exposition.

3.1.2. Reduction of the system

We now apply a reduction procedure due to Marle [Mar95,Mar98] to eliminate the semiholonomic constraint $\dot{x} + x\dot{\theta} = 0$. This reduction procedure consists of eliminating the Lagrange multipliers of a (in general nonholonomically) constrained system by finding a submanifold (the constraint submanifold) endowed with an almost Poisson structure and a Hamiltonian on it in such a way that the dynamics of this almost Poisson dynamical system coincides with the dynamics of the original constrained system. There are several equivalent constructions (see [vdSMa94,Cual95,Mar95,Blal96,Snia01], and references therein) to handle these constraints. It was shown in [vdSMa94] that this almost Poisson structure is actually Poisson if and only if the constraints are semiholonomic.

Let $Q = \mathbb{R} \times S^1$ be the configuration space and $L(x, \theta, \dot{x}, \dot{\theta}) = \frac{1}{2}(\dot{x}^2 + \dot{\theta}^2)$ the Lagrangian of the system subjected to the constraint $\dot{x} + x\dot{\theta} = 0$. Since the Lagrangian L is hyperregular, the Legendre transform $\mathbb{F}L : TQ \to T^*Q$ is an isomorphism that we use to associate a Hamiltonian function $H \in C^\infty(T^*Q)$ to the system. The image by $\mathbb{F}L$ of the constraint submanifold in TQ gives the constraint submanifold P on T^*Q which consists of the points $P = \{(x, \theta, p_x, p_\theta) \in T^*Q \mid p_x + xp_\theta = 0\}$. Let $\mathcal{D} \subset T(T^*Q)$ be the so called *constraint distribution* defined by $\mathcal{D}(z) := T_z P$ for every z in P. D'Alembert's principle provides a prescription to modify the original unconstrained Hamiltonian flow in order to construct a new vector field whose integral curves lie in P. Indeed, let $X_H|_P$ be the restriction of the original Hamiltonian flow to the points in P and let X_D be the modified vector field whose integral curves describe the dynamics of the nonholonomically constrained system. The works by Marle quoted above ensure that, under certain regularity conditions satisfied in this example, the difference $X_W = X_H|_P - X_D$ of these two vector fields, is a section of a subbundle $\mathcal{W}$ of $T_P(T^*Q)$ that satisfies $T_P(T^*Q) = \mathcal{W} \oplus \mathcal{D}$ and that is uniquely determined by D'Alembert's principle. In such a situation, every Hamiltonian vector field can be decomposed in a unique way as $X_H|_P = X_D + X_W$ and X_D describes the dynamics of the constrained system. Marle also shows that there exists an almost Poisson structure on P with almost Poisson tensor $B : T^*P \times T^*P \longrightarrow \mathbb{R}$, for which $X_D = B^\sharp \mathbf{d}H|_P$, where $B^\sharp : T^*P \to TP$ is the canonical vector bundle isomorphism associated to B.

In our example, $\mathcal{D}(x, \theta, p_x, p_\theta) = \operatorname{span}\{(1, 0, -p_\theta, 0), (0, 1, 0, 0), (0, 0, -x, 1)\}$ and $\mathcal{W}(x, \theta, p_x, p_\theta) = \operatorname{span}\{(0, 0, 1, x)\}$. An explicit expression for the almost Poisson structure (see [OrPl04]) can be given by using the natural projection map onto the $\mathcal{D}$ factor. After some computations this almost Poisson tensor takes the form

$$B(x, \theta, p_\theta) = \begin{pmatrix} 0 & 0 & \frac{-x}{1+x^2} \\ 0 & 0 & \frac{1}{1+x^2} \\ \frac{x}{1+x^2} & \frac{-1}{1+x^2} & 0 \end{pmatrix},$$

where the three-tuples (x, θ, p_θ) are used to coordinatize the points $(x, \theta, -xp_\theta, p_\theta) \in P$ and the restricted Hamiltonian is given by $H|_P(x, \theta, p_\theta) = \frac{1}{2}(1 + x^2)p_\theta^2$. Notice that this tensor is Poisson since the constraint is integrable. The equations of motion are

$$\dot{x} = -xp_\theta, \qquad \dot{\theta} = p_\theta, \qquad \text{and} \qquad \dot{p}_\theta = \frac{x^2 p_\theta^2}{1 + x^2}.$$

3.1.3. Equilibria, relative equilibria, and their stability

Notice that every point of the form $z = (x, \theta, 0)$ is an equilibrium of the system. If we first compute the linearization of the dynamical system at those equilibria we obtain the family of matrices

$$\begin{pmatrix} 0 & 0 & -x \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

which have three zero eigenvalues and are not diagonalizable. This implies that the system is linearly unstable at those equilibria (which does not imply either Lyapunov stability or instability).

To apply Theorem 2.5, we first need to find conserved quantities for the Hamiltonian flow. In this case we can use the Hamiltonian and the local Casimir function given by $C(x, \theta, p_\theta) = xe^{\theta}$. Let L be the function defined by $L := \lambda_0 H + \lambda_1 C$. If we set $\lambda_0 = 1$ and $\lambda_1 = 0$ we have that $\mathbf{d}L(z) = 0$. The subspace $W = \ker \mathbf{d}H(z) \cap \mathbf{d}C(z)$ is given by $W = \operatorname{span}\{(x, -1, 0), (0, 0, 1)\}$ and the restricted Hessian

$$\mathbf{d}^2 L(z)\Big|_{W \times W} = \begin{pmatrix} 0 & 0 \\ 0 & (1 + x^2) \end{pmatrix}$$

is not positive definite since it has a zero eigenvalue. The stability of the equilibrium $z = (0, \theta, 0)$ can be analyzed by using the fact that the submanifold S consisting of the points of the form $(0, \theta, p_\theta)$ is such that its vanishing ideal $\mathcal{I}(S)$ is a Poisson ideal and hence S is Poisson reducible. Indeed, if (θ, p_θ) are coordinates on S, the reduced bracket $\{\cdot, \cdot\}_S$ takes the form $\{\theta, p_\theta\}_S = 1$ and the reduced Hamiltonian is $h(\theta, p_\theta) = \frac{1}{2} p_\theta^2$. This reduced system describes a free one dimensional particle. The equilibrium $z = (0, \theta, 0)$ of the original system drops to an equilibrium at the point $(\theta, 0)$ which is clearly unstable. In particular, this implies the instability of the original equilibrium $(0, \theta, 0)$.

We now study the stability of the relative equilibria with respect to the circle symmetry of the system given by the action $\psi \cdot (x, \theta, p_\theta) = (x, \theta + \psi, p_\theta)$. This action is canonical and the system can be Poisson reduced. The reduced manifold is $\mathbb{R}^2$. If we denote by (x, p_θ) the elements of the reduced space, the reduced Poisson bracket is determined by the relation $\{x, p_\theta\} = -x/(1+x^2)$ and the reduced Hamiltonian is $h(x, p_\theta) = \frac{1}{2}(1 + x^2) p_\theta^2$. Hamilton's equations for h are $\dot{x} = -x p_\theta$, $\dot{p}_\theta = x^2 p_\theta/(1 + x^2)$. Thus the equilibria are given by the family of points satisfying $x p_\theta = 0$. The linearization of the Hamiltonian vector field at these equilibria is given by the matrix

$$\begin{pmatrix} -p_\theta & -x \\ 0 & 0 \end{pmatrix},$$

which has a positive eigenvalue if $p_\theta < 0$, in which case the system is Lyapunov unstable at the points $(0, p_\theta)$. This obviously implies that the unreduced system exhibits nonlinearly unstable relative equilibria.

If $p_\theta > 0$ the linearization does not imply neither stability nor instability. However, note that in this case, the linearization has a negative eigenvalue with eigenvector $v = (1, 0)$ that will be useful when searching for a Lyapunov function (see Remark 2.10). In order to study the nonlinear stability of these relative equilibria, we notice that the only available conserved quantity is the reduced Hamiltonian whose derivative $\mathbf{d}h(x, p_\theta) = (xp_\theta^2, (1 + x^2)p_\theta) = (0, 0)$ if and only if $p_\theta = 0$. In that case

$$\mathbf{d}^2 h(x, 0) = \begin{pmatrix} 0 & 0 \\ 0 & (1 + x^2) \end{pmatrix}$$

and hence we cannot conclude either stability or instability. However, in this particular case instability can be concluded just by looking at the phase portrait for the vector field. For points of the form $(0, p_\theta)$ the derivative of the Hamiltonian does not vanish and hence the only way to apply Theorem 2.5 consists of finding a function F satisfying at least one of the hypotheses (i) or (ii); $F(x, p_\theta) = x^2/2$ is one such function since $\{x^2, h\} = -2x^2 p_\theta$, $\{x^4, h\} = -4x^4 p_\theta$, and p_θ is assumed to be positive. Consequently, the hypothesis (i) is obviously satisfied. With this choice, the subspace $W = \ker \mathbf{d}h(0, p_\theta) = \operatorname{span}\{(1, 0)\}$ and $\mathbf{d}^2 F(0, p_\theta)|_{W\times W} = 1 > 0$. Consequently, the equilibria of the form $(0, p_\theta)$ with $p_\theta > 0$ are Lyapunov stable and even though they are not asymptotically stable, there exists an open neighborhood V of $(0, p_\theta)$ such that $F_t(V) \subset F_s(V)$, whenever $t > s$, that is, they are weakly asymptotically stable.

Finally, it is easy to conclude that the equilibria on the form $(x, 0)$ are unstable just by looking at the phase portrait of the system.

3.2. Two coupled spinning wheels

Consider two vertical weightless wheels with radii R and r satisfying $R > r$ and $R/r \in \mathbb{R} \setminus \mathbb{Q}$. We attach to the edges of each of these wheels two point masses M and m (Fig. 1). This simple system has as configuration space Q the torus $\mathbb{T}^2$ that we coordinatize with the angles (θ, φ). The Lagrangian of this system in these coordinates is $L = \frac{1}{2}(MR^2\dot{\theta}^2 + mr^2\dot{\varphi}^2) + MR \cos\theta + mr \cos\varphi$. Assume now that we couple the rotations of the two wheels with a belt. This mechanism imposes on the systems a semiholonomic constraint that can be expressed as $R\dot{\theta} - r\dot{\varphi} = 0$. In order to give a description of the constrained system we first express the original system in the Hamiltonian setting by using the Legendre transform. The phase space P is in this case the cotangent bundle $T^*\mathbb{T}^2 \simeq \mathbb{T}^2 \times \mathbb{R}^2$ with coordinates $(\theta, \varphi, p_\theta, p_\varphi)$, endowed with the canonical symplectic form. The Hamiltonian function is

$$H = \frac{1}{2}\left(\frac{p_\theta^2}{MR^2} + \frac{p_\varphi^2}{mr^2}\right) - MR \cos\theta - mr \cos\varphi.$$

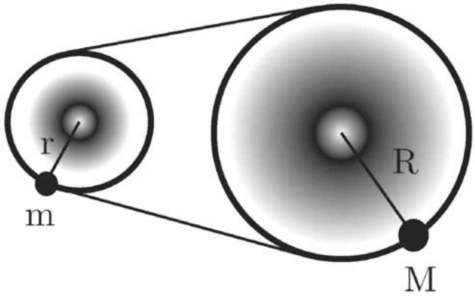


Fig. 1. Two coupled spinning wheels.

The constraint submanifold is given by the points $(\theta, \varphi, p_\theta, p_\varphi)$ that satisfy $p_\varphi = mrp_\theta/MR$, which can be identified with $\mathbb{T}^2 \times \mathbb{R}$ with coordinates (θ, φ, p).

We now apply the reduction procedure in [BaSn93] in order to find a bracket on the constraint submanifold that is actually Poisson since the constraint is semiholonomic. This bracket is given by the constant Poisson tensor:

$$B(\theta, \varphi, p) = \begin{pmatrix} 0 & 0 & r \\ 0 & 0 & R \\ -r & -R & 0 \end{pmatrix}.$$

The reduced Hamiltonian function is

$$h(\theta, \varphi, p) = \frac{p^2}{2k} - MR \cos\theta - mr \cos\varphi,$$

where k is a real positive constant depending on the parameters of the problem given by the expression

$$k = \frac{m+M}{4MmR^2r^2} - \frac{(m-M)^2m^2M^2}{4m^2M^2R^2r^2(m+M)}.$$

This Poisson system has a local Casimir given by the locally defined function $C(\theta, \varphi, p) = R\theta - r\varphi$. The equations of motion of the system are given by

$$\dot{\theta} = r\frac{p}{k}, \qquad \dot{\varphi} = R\frac{p}{k}, \qquad \text{and} \qquad \dot{p} = -rRM \sin\theta - mrR \sin\varphi.$$

The equilibria of this system are the points of the set $S = \{(\theta, \varphi, 0) \mid M \sin\theta + m \sin\varphi = 0\}$ that can be described as a one-parameter family given by the curve $\varphi = -\sin^{-1}\left(\frac{M \sin\theta}{m}\right)$, $\theta \in [-\theta_c, \theta_c]$, where θ_c is given by $\theta_c = \sin^{-1}(\frac{m}{M})$. In order to study the nonlinear stability of such equilibria we compute $\lambda_0\, \mathbf{d}h(z) + \lambda_1\, \mathbf{d}C(z) = 0$, with $z = (\theta, \varphi, 0) \in S$. This equation can be solved by taking $\lambda_1 = -\lambda_0 M \sin\theta$. In this case $W = \ker \mathbf{d}C(z) \cap \ker \mathbf{d}h(z) = \operatorname{span}\{(r, R, 0), (0, 0, 1)\}$. Finally, it is easy to see that $\mathbf{d}^2(\lambda_0 h + \lambda_1 C)(z)|_{W\times W} > 0$ if and only if $Mr\cos\theta + mR\cos\varphi > 0$. In particular, the point $z = (0, 0, 0)$ is always nonlinearly stable, as expected, and the point $z = (0, \pi, 0)$ is stable if $\frac{M}{R} > \frac{m}{r}$.

4. Nonlinear stability via topological methods

In [Paal04] topology based tools have been developed that provide sufficient conditions for the Lyapunov stability of Poisson equilibria. One of the main achievements in [Paal04] is the discovery of a space related to the topology of the symplectic foliation of the Poisson manifold (see (4.3) below) on which the extremality of the Hamiltonian suffices to conclude stability. In this section we will study under which circumstances the topological criteria in [Paal04] can be expressed in terms of local continuous Casimir functions and hence there is an equivalence with the energy-Casimir method. To be more explicit, we will seek the correspondence between the topological approach of [Paal04] and a generalization of the energy-Casimir method that requires only continuity of the functions involved and that is based on the following general lemma.

Lemma 4.1. *Let $X \in \mathfrak{X}(P)$ be a smooth vector field on the finite dimensional manifold P and $z_{\rm e} \in P$ an equilibrium point. If there exists locally defined continuous conserved quantities $C_0, \ldots, C_k \in C^0(U)$ of the flow F_t such that $\bigcap_{i=0}^{k} C_i^{-1}(C_i(z_{\rm e})) = \{z_{\rm e}\}$ then the equilibrium $z_{\rm e}$ is Lyapunov stable.*

Proof. Consider the function $L(z) = (C_0(z) - C_0(z_{\rm e}))^2 + \cdots + (C_k(z) - C_k(z_{\rm e}))^2$. The hypothesis $\bigcap_{i=0}^{k} C_i^{-1}(C_i(z_{\rm e})) = \{z_{\rm e}\}$ ensures that L is a positive function that takes the zero value only at the point $z_{\rm e}$. In particular, the sets of the form $L^{-1}([0, \varepsilon))$, $\varepsilon > 0$, form a fundamental system of neighborhoods in the manifold topology of P at the point $z_{\rm e}$. Consequently, for any open neighborhood U of $z_{\rm e}$ there exists an $\varepsilon > 0$ such that $L^{-1}([0, \varepsilon]) \subset U$. Since the level set $L^{-1}(\varepsilon)$ is invariant by the flow F_t of X, the Lyapunov stability of $z_{\rm e}$ follows. □

Remark 4.2. In the same way in which in Theorem 2.5 we could take advantage of nonconserved quantities in concluding the stability of a given equilibrium, Lemma 4.1 can be reformulated as:

> Let $X \in \mathfrak{X}(P)$ be a smooth vector field on P and $z_{\rm e} \in P$ an equilibrium. Let $C_0, \ldots, C_k \in C^0(U)$ be continuous functions locally defined around $z_{\rm e}$ such that $X[C_i^2] \leqslant 0$, $i \in \{0, \ldots, k\}$. If $\bigcap_{i=0}^{k} C_i^{-1}(C_i(z_{\rm e})) = \{z_{\rm e}\}$ then the equilibrium $z_{\rm e}$ is Lyapunov stable.

Any continuous function $C \in C^0(U)$, with U an open subset of P, such that C is constant on the symplectic leaves of $(U, \{\cdot,\cdot\}|_U)$ is called a *local continuous Casimir* of $(P, \{\cdot, \cdot\})$. The choice of terminology is justified by the fact that if such a function C happens to be differentiable then it is an actual Casimir of $(U, \{\cdot, \cdot\}|_U)$. It is worth noticing that the local continuous Casimirs are the (continuous) first integrals of the foliation of $(U, \{\cdot, \cdot\}|_U)$ by its symplectic leaves.

Corollary 4.3 (*Continuous energy-Casimir method*). *Let $(P, \{\cdot, \cdot\}, H)$ be a Poisson dynamical system and $z_e \in P$ an equilibrium point of the Hamiltonian vector field X_H. Let $S_{z_e} \subset P$ be the common level set of local continuous Casimir functions around z_e. If*

$$H^{-1}(H(z_e)) \cap S_{z_e} = \{z_e\} \tag{4.1}$$

then the equilibrium z_e is Lyapunov stable. This statement remains true if H is replaced by any continuous conserved quantity of the flow of X_H.

Our goal is to establish sufficient conditions under which this corollary coincides with the topological stability criterion in [Paal04] that we now recall. We start by introducing the necessary notation. Let (X, τ) be a topological space and $x \in X$ an arbitrary point. We define the set $T_2(x) \subset X$ as

$$\begin{aligned} T_2(x) := \{y \in X \mid U_x \cap U_y \neq \emptyset \text{ for any two open neighborhoods} \\ U_x,\, U_y \text{ of } x \text{ and } y\}. \end{aligned} \tag{4.2}$$

Let $A \subset X$ be an arbitrary subset. We define

$$T_2(A) := \bigcup_{x \in A} T_2(x).$$

Notice that if $y \in T_2(x)$ then $x \in T_2(y)$. Also, a topological space (X, τ) is Hausdorff if and only if $T_2(x) = x$ for every $x \in X$. Hence the T_2 sets measure how far a topological space is from being Hausdorff.

Suppose now that P is a smooth Hausdorff and paracompact finite dimensional manifold and D is a smooth and integrable generalized distribution on P. Let $\pi_D : P \to P/D$ be the projection onto the leaf space of the distribution D. The map π is continuous and open when P/D is endowed with the quotient topology. Define

$$\overline{T}_2(x) = \pi_D^{-1}\left(T_2\left(\pi_D(x)\right)\right), \quad x \in P \tag{4.3}$$

and, more generally,

$$\overline{T}_2^U(x) = \pi_{D|_U}^{-1}\left(T_2\left(\pi_{D|_U}(x)\right)\right), \quad x \in P, \tag{4.4}$$

where U is an open neighborhood of $x \in P$ and $\pi_{D|_U} : U \to U/D|_U$ is the projection onto the leaf space of the restriction $D|_U$ of D to U.

We now focus on the particular case when P is a Poisson manifold with bracket $\{\cdot,\cdot\}$. Let $\mathcal{E}$ be the corresponding characteristic distribution and $\pi : P \to P/\{\cdot,\cdot\}$ the projection onto the space of symplectic leaves $P/\{\cdot,\cdot\} := P/\mathcal{E}$.

Theorem 4.4 (*Topological energy-Casimir method; Patrick et al.* [Paal04]). *Let $(P, \{\cdot, \cdot\}, H)$ be a Poisson dynamical system and $z_e \in P$ an equilibrium point for the Hamiltonian vector field X_H. If there is an open neighborhood $U \subset P$ of z_e such that*

$$H^{-1}(H(z_e)) \cap \overline{T}_2^U(z_e) = \{z_e\} \tag{4.5}$$

then the equilibrium z_e is Lyapunov stable. This statement remains true if H is replaced by any continuous conserved quantity of the flow of X_H that takes values in a Hausdorff space.

In view of expressions (4.1) and (4.5) we would like to know under what circumstances the set $\overline{T}_2^U(z_e)$ can be obtained by looking at the level sets of local continuous Casimir functions thereby rendering the statements of Corollary 4.3 and Theorem 4.4 equivalent.

The first point that we have to emphasize is that this is, in general, not possible. The following example, that we owe to James Montaldi, shows that, in general, we cannot find enough local Casimir functions to be able to write the set $\overline{T}_2^U(z_e)$ as the common level set of local continuous Casimir functions, no matter how much we shrink the neighborhood U. Let $\mathbb{R}^3$ and $f(x, y, z) = x^2 + y^2 - z^2$. Consider the Poisson structure $\{\cdot,\cdot\}$ determined by $\{x, y\} = f^2$, $\{y, z\} = 2yzf$, and $\{x, z\} = -2xzf$. In order to describe the symplectic leaves of $(\mathbb{R}^3, \{\cdot,\cdot\})$ (see Fig. 2) notice first that the function f is a factor and hence the Poisson tensor vanishes on the cone $f = 0$. Consider now all the spheres through the origin and tangent to the *OXY* plane (and hence centered on the *OZ*-axis) and cut them with the cone $f = 0$. Each of these spheres contains the following symplectic leaves: the sphere intersected with the points (x, y, z) such that $f(x, y, z) > 0$ (two-dimensional leaf), the sphere intersected with the points (x, y, z) such that $f(x, y, z) < 0$ (two-dimensional leaf), and the points such that $f(x, y, z) = 0$ (zero dimensional leaves). It is clear from this description that there are no nonconstant continuous local Casimir functions near the origin. Nevertheless, for any neighborhood U of the origin $\overline{T}_2^U(0, 0, 0) = \{(x, y, z) \mid f(x, y, z) \geqslant 0\}$, that is, the closed exterior of the cone, which in this case is strictly included in $C_U^{-1}(C_U(0, 0, 0)) = U$.

Even though the previous example shows that the set $\overline{T}_2^U(z_e)$ does not coincide in general with the common level set of local continuous Casimir functions one can easily prove that at least one inclusion holds true. The natural context to present most of the results in this section is that of generalized foliations of smooth manifolds. Consequently, we will prove our statements in that category and we will obtain the Poisson case as a corollary by applying the theorems to the generalized foliation of the Poisson manifold by its symplectic leaves.

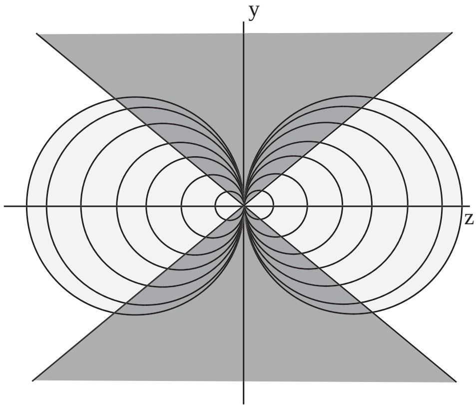
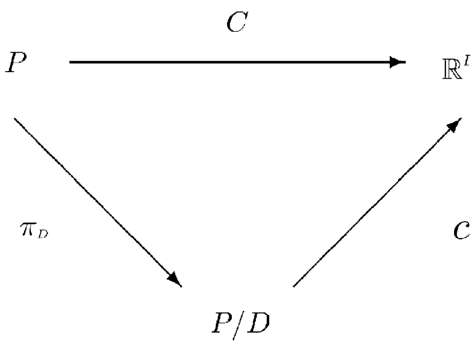


Fig. 2. Symplectic leaves of Montaldi's example of a Poisson manifold that does not have local Casimirs around the origin. The shadowed area represents the set $\overline{T}_2^U(0,0,0)$. The picture is a section of the three dimensional figure through the *OYZ* plane.

Lemma 4.5. *Let P be a smooth finite dimensional manifold and D a smooth integrable generalized distribution on P. Let $\pi_D : P \to P/D$ be the projection onto the leaf space of the distribution D and $\overline{T}_2$ the symbol defined in* (4.3). *Let $C_i \in C^0(P)$, $i \in I$, be a set of continuous functions that are constant on the integral leaves of D* (*that is, first integrals of D*). *Then for any $z \in P$*

$$\overline{T}_2(z) \subset \bigcap_{i \in I} C_i^{-1}(C_i(z)). \tag{4.6}$$

Proof. Let $C : P \to \mathbb{R}^I$ be the function defined by $C(z) := (C_i(z))_{i \in I}$. If we endow $\mathbb{R}^I$ with the product topology (not the box topology!) then the continuity of the first integrals C_i, $i \in I$, implies that C is continuous. The projection $\pi : P \to P/D$ is an open map when P/D is endowed with the quotient topology. Given that C is constant on the integral leaves of D it drops to a map $c : P/D \to \mathbb{R}^I$ that closes the diagram



The continuity of C and the openness and surjectivity of π_D imply that c is also continuous. In order to prove (4.6) it suffices to show that if $m \in \overline{T}_2(z)$ then $C(m) = C(z)$. By contradiction, suppose that $C(m) \neq C(z)$. Since $\mathbb{R}^I$ is a Hausdorff topological space there are open neighborhoods $V_{C(m)}$ and $V_{C(z)}$ of m and z, respectively, such that $V_{C(m)} \cap V_{C(z)} = \emptyset$. As c is continuous the sets $c^{-1}(V_{C(m)})$ and $c^{-1}(V_{C(z)})$ are open neighborhoods of $\pi_D(m)$ and $\pi_D(z)$, respectively. Also, since by hypothesis $m \in \overline{T}_2(z)$, we have that $c^{-1}(V_{C(m)}) \cap c^{-1}(V_{C(z)}) \neq \emptyset$. However, by construction, we also have that $c^{-1}(V_{C(m)}) \cap c^{-1}(V_{C(z)}) = c^{-1}(V_{C(m)} \cap V_{C(z)}) = c^{-1}(\emptyset) = \emptyset$, which is a contradiction. □

The rest of this section is dedicated to the description of two situations where the inclusion (4.6) is an equality and hence local continuous Casimir functions characterize the $\overline{T}_2$-sets. We start with a couple of preliminary general results.

Definition 4.6. Let (X, τ) be a topological space. We say that (X, τ) is *T_2-idempotent* when $T_2(T_2(x)) = T_2(x)$, for any $x \in X$.

Lemma 4.7. *Let (X, τ) be a T_2-idempotent topological space. Then*

(i) *The relation $\mathcal{R}_{T_2}$ on X defined by $x\mathcal{R}_{T_2}y$ if and only if $y \in T_2(x)$ is an equivalence relation on X.*
(ii) *The following statements are equivalent*:
   1. $y \notin T_2(x)$.
   2. $T_2(x) \neq T_2(y)$.
   3. $T_2(x) \cap T_2(y) = \emptyset$.
   4. *There exist open neighborhoods U_x, U_y of x and y, respectively, such that $T_2(U_x) \cap T_2(U_y) = \emptyset$.*
(iii) *If the projection $\pi_{T_2} : X \longrightarrow X/\mathcal{R}_{T_2}$ onto the space of equivalence classes endowed with the quotient topology is an open map then $X/\mathcal{R}_{T_2}$ is a Hausdorff topological space.*

Proof. (i) The definition of the T_2 set implies that $x\mathcal{R}_{T_2}x$ for any $x \in X$ and that $x\mathcal{R}_{T_2}y$ if and only if $y\mathcal{R}_{T_2}x$. In order to prove transitivity of $\mathcal{R}_{T_2}$ let $x, y, z \in X$ be such that $x\mathcal{R}_{T_2}y$ and $y\mathcal{R}_{T_2}z$. By the very definition of the T_2 set, it is clear that for any two subsets $A, B \subset X$ such that $A \subset B$ we have that $T_2(A) \subset T_2(B)$. In particular, the condition $x \in T_2(y)$ implies that $T_2(x) \subset T_2(T_2(y)) = T_2(y)$. By reflexivity we have that $T_2(y) \subset T_2(x)$ and hence $T_2(x) = T_2(y)$ which implies that $T_2(x) = T_2(y) = T_2(z)$ and hence $x\mathcal{R}_{T_2}z$.

(ii) If $T_2(x) = T_2(y)$ then $y \in T_2(y) = T_2(x)$. This proves the implication $1 \Rightarrow 2$. The implication $2 \Rightarrow 1$ was already proved in the first part of the lemma. In order to prove $2 \Rightarrow 3$ suppose that there exists a point $z \in T_2(x) \cap T_2(y)$. Then using the T_2 idempotency as we did in the proof of the first part of the lemma we obtain that $T_2(x) = T_2(z) = T_2(y)$, which contradicts the hypothesis. To show $3 \Rightarrow 4$, assume that $T_2(x) \cap T_2(y) = \emptyset$. Then, in particular, $y \notin T_2(x)$ and hence there exist open neighborhoods U_x and U_y of x and y, respectively, such that $U_x \cap U_y = \emptyset$. Since

U_x and U_y are open neighborhoods of each of their points, it follows that for every $a_x \in U_x$ and $a_y \in U_y$ the element $a_x \notin T_2(a_y)$. Using the implication $1 \Rightarrow 3$ that we have already proved, this shows that $T_2(a_x) \cap T_2(a_y) = \emptyset$ and hence

$$T_2(U_x) \cap T_2(U_y) = \left(\bigcup_{a_x \in U_x} T_2(a_x) \right) \bigcap \left(\bigcup_{a_y \in U_y} T_2(a_y) \right)$$
$$= \bigcup_{a_x \in U_x, a_y \in U_y} \left(T_2(a_x) \bigcap T_2(a_y) \right) = \emptyset.$$

Finally, the implication 4⇒2 is straightforward.

(iii) Notice first that for every subset $A \subset X$, we have that $\pi_{T_2}^{-1}(\pi_{T_2}(A)) = T_2(A)$. Let $\rho, \sigma \in X/\mathcal{R}_{T_2}$ be two points such that $\rho \neq \sigma$ and let x and y be two points in X such that $\pi_{T_2}(x) = \rho$ and $\pi(y) = \sigma$. Since $T_2(x) \neq T_2(y)$ there exist, by part (ii), two open neighborhoods V_x and V_y of x and y, respectively, such that $\emptyset = T_2(V_x) \cap T_2(V_y) = \pi_{T_2}^{-1}(\pi_{T_2}(V_x)) \cap \pi_{T_2}^{-1}(\pi_{T_2}(V_y)) = \pi_{T_2}^{-1}(\pi_{T_2}(V_x) \cap \pi_{T_2}(V_y))$. Applying π_{T_2} to both sides of this equality we obtain that $\pi_{T_2}(V_x) \cap \pi_{T_2}(V_y) = \emptyset$. Since $\pi_{T_2}(V_x)$ and $\pi_{T_2}(V_y)$ are, by the openness of π_{T_2}, open neighborhoods of the points ρ and σ, respectively, the claim follows. □

Suppose now that P is a smooth Hausdorff and paracompact finite dimensional manifold and D is a smooth and integrable generalized distribution on P. Let $\pi_D : P \to P/D$ be the projection onto the leaf space of the distribution D and $\overline{T}_2$ the symbol defined in (4.3). Notice that since π_D is surjective, we have

$$\pi_D(\overline{T}_2(x)) = T_2\left(\pi_D(x)\right), \quad \text{for any } x \in P. \tag{4.7}$$

We will say that the pair (P, D) is *$\overline{T}_2$-idempotent* when $\overline{T}_2(\overline{T}_2(x)) = \overline{T}_2(x)$, for any $x \in P$. Notice that since the sets $\overline{T}_2(x)$ are D-saturated (they are unions of leaves of D), we can conclude, using (4.7), that P is $\overline{T}_2$-idempotent if and only if P/D is T_2-idempotent. With this remark in mind the previous lemma can be easily adapted to the symbol $\overline{T}_2$.

Lemma 4.8. *Let P be a smooth Hausdorff paracompact finite dimensional manifold and D a smooth integrable generalized distribution on P. Let $\pi_D : P \to P/D$ be the projection onto the leaf space of the distribution D and $\overline{T}_2$ the symbol defined in* (4.3). *Suppose that (P, D) is $\overline{T}_2$-idempotent. Then*:

(i) *The relation $\mathcal{R}_{\overline{T}_2}$ on P defined by $x \mathcal{R}_{\overline{T}_2} y$ if and only if $y \in \overline{T}_2(x)$ is an equivalence relation.*

(ii) *The following properties are equivalent*:

1. $y \notin \overline{T}_2(x)$.
2. $\overline{T}_2(x) \neq \overline{T}_2(y)$.

3. $\overline{T}_2(x) \cap \overline{T}_2(y) = \emptyset$.
4. *There exist open neighborhoods* V_x, V_y *of* x *and* y*, respectively, such that* $\overline{T}_2(V_x) \cap \overline{T}_2(V_y) = \emptyset$.

(iii) *If the projection* $\pi_{\overline{T}_2} : P \longrightarrow P/\mathcal{R}_{\overline{T}_2}$ *is an open map then the quotient space* $P/\mathcal{R}_{\overline{T}_2}$ *is a Hausdorff topological space.*

Proof. (i) Only transitivity needs to be proved. Let $x, y, z \in P$ be such that $x\mathcal{R}_{\overline{T}_2}y$ and $y\mathcal{R}_{\overline{T}_2}z$. By definition, $\pi_D(x)\mathcal{R}_{T_2}\pi_D(y)$ and $\pi_D(y)\mathcal{R}_{T_2}\pi_D(z)$. Since the $\overline{T}_2$-idempotency of (P, D) is equivalent to the T_2-idempotency of P/D, Lemma 4.7 guarantees that $\pi_D(x)\mathcal{R}_{T_2}\pi_D(z)$ and hence $\pi_D(x) \in T_2(\pi_D(z))$. Consequently, $x \in \pi_D^{-1}(T_2(\pi_D(z))) = \overline{T}_2(z)$ and thus $z\mathcal{R}_{\overline{T}_2}x$.

In order to prove parts (ii) and (iii) it suffices to mimic the corresponding implications in Lemma 4.7 but, this time, keeping in mind that the projection $\pi_{\overline{T}_2} : P \to P/\mathcal{R}_{\overline{T}_2}$, $\pi_{\overline{T}_2} = \pi_{T_2} \circ \pi_D$, is just the composition of two projection maps and that π_D is an open map. $\square$

Theorem 4.9. *Let* P *be a smooth Hausdorff paracompact finite dimensional manifold and* D *a smooth integrable generalized distribution on* P*. Let* $\pi_D : P \to P/D$ *be the projection onto the leaf space of the distribution* D *and* $\overline{T}_2$ *the symbol defined in* (4.3)*. Suppose that* (P, D) *is* $\overline{T}_2$*-idempotent and that* π_{T_2} (*and hence* $\pi_{\overline{T}_2}$) *is open. Then the continuous first integrals of* D *separate the* $\overline{T}_2$ *sets. In this situation, for any* $z \in P$*, there exist continuous first integrals* $\{C_i\}_{i\in I} \subset C^0(P)$ *of* D *such that*

$$\overline{T}_2(z) = \bigcap_{i\in I} C_i^{-1}(C_i(z)). \tag{4.8}$$

Proof. Since P is by hypothesis paracompact, so are the quotient spaces P/D and $P/\mathcal{R}_{\overline{T}_2}$. The hypothesis on the $\overline{T}_2$-idempotency of (P, D) implies, by Lemma 4.8, that the quotient space $P/\mathcal{R}_{\overline{T}_2}$ is also Hausdorff. Since a Hausdorff paracompact space is normal, Urysohn's Lemma guarantees the existence of continuous functions f on $P/\mathcal{R}_{\overline{T}_2}$ that separate two given distinct points. The pull back $f \circ \pi_{\overline{T}_2} \in C^0(P)$ is a first integral of D. The family of functions of the form $f \circ \pi_{\overline{T}_2}$ where $f : P/\mathcal{R}_{\overline{T}_2} \to \mathbb{R}$ is a continuous function that separates two arbitrary points, is the family of continuous first integrals of D in the statement of the theorem.

In order to prove the identity (4.8) it suffices to reproduce the proof of Lemma 4.5, taking this time the function $C : P \to \mathbb{R}^I$ whose components are the continuous first integrals of D that separate the $\overline{T}_2$ sets and whose existence we just proved. $\square$

Remark 4.10. The two hypotheses in the statement of this result, that is, the $\overline{T}_2$-idempotency and the openness of the projection $\pi_{\overline{T}_2}$, are independent. Indeed, consider the foliation of the Euclidean plane $\mathbb{R}^2$ by the integral curves of the vector field $\phi(x)\partial/\partial x$, where ϕ is a smooth function satisfying $\phi(x) = 0$, for $x \leqslant 0$, and $\phi(x) > 0$, for $x > 0$. In this situation $\overline{T}_2(x, y) = \{(x, y)\}$, when $x < 0$, and $\overline{T}_2(x, y) = \{(x, y) \in \mathbb{R}^2 \mid x \leqslant 0\}$, if $x \geqslant 0$. In this situation, we obviously have $\overline{T}_2$-idempotency. However, the projection $\pi_{\overline{T}_2} : \mathbb{R}^2 \to \mathbb{R}^2/\mathcal{R}_{\overline{T}_2}$ is not open. Indeed, the saturation $\pi_{\overline{T}_2}^{-1}(\pi_{\overline{T}_2}(U)) = \{(x, y) \in \mathbb{R}^2 \mid x \geqslant 0\}$ of the open set $U = \{(x, y) \in \mathbb{R}^2 \mid x > 0\}$ is closed, which is not compatible with $\pi_{\overline{T}_2}$ being open.

The following result provides another sufficient condition for the conclusion of Theorem 4.9 to hold.

Theorem 4.11. *Let D be a generalized smooth integrable distribution defined on the second countable finite dimensional manifold P. Suppose that there exist continuous first integrals $C_i \in C^0(P)$, $i \in I$, of the foliation induced by D that separate its regular leaves. Additionally, assume that the map $C : P \to \mathbb{R}^I$ defined by $C(z) := (C_i(z))_{i\in I}$, $z \in P$, is open onto its image when $\mathbb{R}^I$ is endowed with the product topology. Then for any $z \in P$*

$$\overline{T}_2(z) = \bigcap_{i\in I} C_i^{-1}(C_i(z)).$$

Proof. Notice first that the inclusion

$$\overline{T}_2(z) \subset \bigcap_{i\in I} C_i^{-1}(C_i(z)).$$

is a particular case of (4.6).

In order to prove the converse inclusion let $\pi_D : P \to P/D$ be the projection onto the leaf space and $c : P/D \to \mathbb{R}^k$ the continuous mapping uniquely determined by the relation $c \circ \pi_D = C$. Let $n \in \bigcap_{i\in I} C_i^{-1}(C_i(z))$, that is, $C(n) = C(z)$ and assume that $n \notin \overline{T}_2(z)$. This implies the existence of two open neighborhoods $V_{\pi_D(n)}$ and $V_{\pi_D(z)}$ of $\pi_D(n)$ and $\pi_D(z)$, respectively, such that $V_{\pi_D(n)} \cap V_{\pi_D(z)} = \emptyset$. We will assume for the time being that the leaf $\pi_D(n)$ is regular and will prove that the assumption $n \notin \overline{T}_2(z)$ leads to a contradiction. We will prove later on that the situation in which $\pi_D(n)$ is a singular leaf can be reduced to this case.

If $\pi_D(n)$ is regular, the set $V^{\rm reg}_{\pi_D(n)}$ of regular leaves in $V_{\pi_D(n)}$ is an open dense neighborhood of $\pi_D(n)$ in $V_{\pi_D(n)}$. The openness hypothesis on the map C implies that the set

$$U_{C(z)} := c(V^{\rm reg}_{\pi_D(n)}) \cap c(V_{\pi_D(z)})$$

is an open neighborhood of $C(z)$. Moreover, the continuity of c implies that the sets

$$A := c^{-1}(U_{C(z)}) \cap V^{\rm reg}_{\pi_D(n)} \quad \text{and} \quad B := c^{-1}(U_{C(z)}) \cap V_{\pi_D(z)}$$

are open neighborhoods of $\pi_D(n)$ and $\pi_D(z)$, respectively. Let $\pi_D(z')$ be a regular leaf in B. The construction of B implies that there exists a regular leaf $\pi_D(s) \in V^{\rm reg}_{\pi_D(n)} \subset V_{\pi_D(n)}$ such that $c(\pi_D(z')) = c(\pi_D(s))$. The separation hypothesis on the map C implies that $\pi_D(s) = \pi_D(z') \in V_{\pi_D(n)} \cap V_{\pi_D(z)}$ which is a contradiction.

In order to conclude the proof we need to show that the case in which $\pi_D(n)$ is singular can be reduced to the situation that we just treated. Indeed, take $U_{C(z)} := c(V_{\pi_D(n)}) \cap c(V_{\pi_D(z)})$. By the openness of C, $U_{C(z)}$ is an open neighborhood of $C(z)$. Additionally, the continuity of c implies that the sets $A := c^{-1}(U_{C(z)}) \cap V_{\pi_D(n)}$ and $B := c^{-1}(U_{C(z)}) \cap V_{\pi_D(z)}$ are open disjoint neighborhoods of $\pi_D(n)$ and $\pi_D(z)$, respectively. Let $\pi_D(z')$ be a regular leaf in A. The construction of A implies the existence of a leaf $\pi_D(s) \in V_{\pi_D(z)}$ such that $C(\pi_D(z')) = C(\pi_D(s))$. If we follow the preceding argument, replacing $\pi_D(z')$ by $\pi_D(n)$, $\pi_D(s)$ by $\pi_D(z)$, A by $V_{\pi_D(n)}$, and with $V_{\pi_D(z)}$ playing the same role we also obtain a contradiction with the hypothesis $V_{\pi_D(n)} \cap V_{\pi_D(z)} = \emptyset$. □

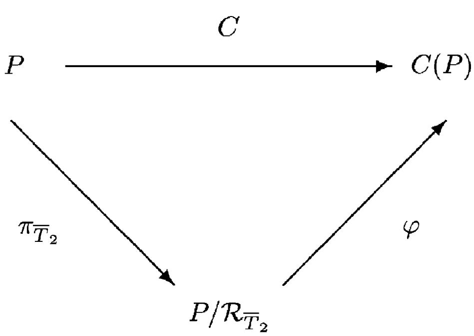
The reader may be wondering how the two sufficient conditions for (4.8) to hold that we presented in the statements of Theorems 4.9 and 4.11 are related. Our next result answers this question.

Proposition 4.12. *Let P be a smooth second countable finite dimensional manifold and D a smooth integrable generalized distribution on P. Suppose that there exist continuous first integrals $C_i \in C^0(P)$, $i \in I$, of the foliation induced by D that separate its regular leaves such that the map $C : P \to C(P) \subset \mathbb{R}^I$ defined by $C(z) := (C_i(z))_{i \in I}$, $z \in P$, is open onto its image when $\mathbb{R}^I$ is endowed with the product topology. Then (P, D) is $\overline{T}_2$-idempotent and $\pi_{\overline{T}_2} : P \to P/\mathcal{R}_{\overline{T}_2}$ is an open map.*

Proof. In the hypotheses of the statement, Theorem 4.11 implies that $\overline{T}_2(z) = C^{-1}(C(z))$, for any $z \in P$. In particular

$$\overline{T}_2(\overline{T}_2(z)) = \overline{T}_2(C^{-1}(C(z))) = \bigcup_{y \in C^{-1}(C(z))} \overline{T}_2(y) = C^{-1}(C(z)) = \overline{T}_2(z),$$

which guarantees that (P, D) is $\overline{T}_2$-idempotent and hence allows us to define an equivalence relation $\mathcal{R}_{\overline{T}_2}$ on P. We will now show that the associated projection to the quotient $\pi_{\overline{T}_2} : P \to P/\mathcal{R}_{\overline{T}_2}$ is open. Let $\varphi : P/\mathcal{R}_{\overline{T}_2} \to C(P)$ be the map defined by $\varphi(\pi_{\overline{T}_2}(z)) := C(z)$, $z \in P$. The equality $\overline{T}_2(z) = C^{-1}(C(z))$, $z \in P$, guarantees that φ is a well defined bijection that makes the diagram



commutative. The continuity and the openness of C imply respectively the continuity and the openness of φ, that is, φ is a homeomorphism. Since $\pi_{\overline{T}_2} = \varphi^{-1} \circ C$, the openness of $\pi_{\overline{T}_2}$ follows. $\square$

Remark 4.13. The converse of the implication in the previous proposition is not true in general. A counterexample to this effect is an irrational foliation of the two-torus. In that particular case the $\overline{T}_2$ set of any point is the entire torus and hence we have $\overline{T}_2$-idempotency with a projection $\pi_{\overline{T}_2} : P \to P/\mathcal{R}_{\overline{T}_2}$ that is obviously open. Nevertheless, the only first integrals of this foliation are the constant functions that do not separate the leaves of the foliation, all of which happen to be regular in this case.

We now collect the results in Theorems 4.9 and 4.11 and in Proposition 4.12 and we apply them to the situation in which P is a Poisson manifold foliated by its symplectic leaves. The following result provides two sufficient conditions for the continuous and topological energy-Casimir methods to coincide.

Theorem 4.14. *Let $(P, \{\cdot, \cdot\})$ be a Poisson (paracompact, second countable, and Hausdorff) manifold. Let $\overline{T}_2$ be the symbol associated to the symplectic foliation of P induced by the Poisson structure $\{\cdot, \cdot\}$.*

(i) *Suppose that there exist continuous Casimir functions $C_i \in C^0(P)$, $i \in I$, that separate the regular symplectic leaves of P such that the map $C : P \to C(P) \subset \mathbb{R}^I$ defined by $C(z) := (C_i(z))_{i\in I}$, $z \in P$, is open onto its image when $\mathbb{R}^I$ is endowed with the product topology. Then P is $\overline{T}_2$-idempotent and $\pi_{\overline{T}_2} : P \to P/\mathcal{R}_{\overline{T}_2}$ is an open map.*

(ii) *If $(P, \{\cdot, \cdot\})$ is $\overline{T}_2$-idempotent and $\pi_{\overline{T}_2} : P \to P/\mathcal{R}_{\overline{T}_2}$ is an open map then there exist continuous Casimir functions $\{C_i\}_{i\in I} \subset C^0(P)$ of $(P, \{\cdot, \cdot\})$ such that for any $z \in P$*

$$\overline{T}_2(z) = \bigcap_{i\in I} C_i^{-1}(C_i(z)).$$

Remark 4.15. As one could expect, the hypotheses of this theorem are not satisfied by Montaldi's example (see Fig. 2). Indeed, in this particular case $\overline{T}_2^U(\overline{T}_2^U(0,0,0)) = U \neq \overline{T}_2^U(0,0,0)$, for any open neighborhood U of the origin $(0,0,0)$.

Remark 4.16. In all the stable examples in Section 4.3 of [Paal04] there exist local casimirs $\{C_i\}_{i\in I}$ so that $\overline{T}_2(z) = \bigcap_{i\in I} C_i^{-1}(C_i(z))$ and hence the continuous energy-Casimir method in Corollary 4.3 suffices to prove stability. We show this explicitly for one of those examples in the following paragraphs.

Example 4.17. Let $(\mathbb{R}^3, \{\cdot, \cdot\}, h)$ be the Poisson dynamical system given by

$$\{f, g\} = \nabla A \cdot (\nabla f \times \nabla g), \qquad A(x, y, z) = (a^2x^2 - y^2)y,$$

where a is a nonzero real constant and $h(x, y, z) = x^2 - y^2 + z^2$. The equations of motion are

$$\dot{x} = 2z(a^2x^2 - 3y^2), \quad \dot{y} = -4a^2xyz, \quad \text{and} \quad \dot{z} = -2a^2x^3 + (6 - 4a^2)xy^2.$$

Notice that the function A is a Casimir of the bracket $\{\cdot, \cdot\}$ and that the points of the form $(0, y, 0)$ and $(0, 0, z)$ are equilibria of the Hamiltonian vector field X_h. We will focus on the stability of the equilibrium at the origin $m = (0, 0, 0)$ that happens to be a singular point of the symplectic foliation of $\mathbb{R}^3$. In order to verify that the hypotheses of Corollary 4.3 are satisfied notice that the map A can be rewritten as $A(x, y, z) = (ax + y)(ax - y)y$ and hence its zero level set (the one containing the equilibrium $(0, 0, 0)$) can be written as the union of three irreducible algebraic varieties V_1, V_2, and V_3 that are the zero level sets of the functions $y, ax - y$, and $ax + y$, respectively. Consequently,

$$\begin{aligned} h^{-1}(0) \cap A^{-1}(0) &= h^{-1}(0) \bigcap (V_1 \cup V_2 \cup V_3) \\ &= (h^{-1}(0) \cap V_1) \cup (h^{-1}(0) \cap V_2) \cup (h^{-1}(0) \cap V_3), \end{aligned} \tag{4.9}$$

which is a single point whenever $|a| < 1$ hence proving the Lyapunov stability of $(0, 0, 0)$. This is so since each of the three intersections on the right-hand side of expression (4.9) coincide with the point m. This statement can be proved by showing that the Hamiltonian restricted to the submanifolds V_1, V_2 and V_3 has a nondegenerate critical point at $(0, 0, 0)$. This is closely related to the *smoothing* of the $\overline{T}_2$ set introduced in [Paal04].

Since the Casimir function A clearly separates the regular symplectic leaves of $(\mathbb{R}^3, \{\cdot, \cdot\})$ and it is an open map, by Theorem 4.14 we can conclude that

$$\overline{T}_2(0, 0, 0) = A^{-1}(A(0, 0, 0))$$

and hence energy-Casimir and T_2-based sufficient stability conditions can be used interchangeably.

It is worth mentioning that the equilibrium $m = (0, 0, 0)$ is unstable for $|a| \geqslant 1$. This can be seen by inspection of the equations of motion. Hence the stability condition $|a| < 1$ is sharp.

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