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Bifurcation of relative equilibria in mechanical systems with symmetry

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Abstract

The relative equilibria of a symmetric Hamiltonian dynamical system are the critical points of the so-called *augmented Hamiltonian*. The underlying geometric structure of the system is used to decompose the critical point equations and construct a collection of implicitly defined functions and reduced equations describing the set of relative equilibria in a neighborhood of a given relative equilibrium. The structure of the reduced equations is studied in a few relevant situations. In particular, a persistence result of Lerman and Singer [Nonlinearity 11 (1998) 1637–1649] is generalized to the framework of Abelian proper actions. Also, a Hamiltonian version of the Equivariant Branching Lemma and a study of bifurcations with maximal isotropy are presented. An elementary example illustrates the use of this approach.

1. Introduction

The systematic analysis of bifurcations of relative equilibria was greatly stimulated about fifteen years ago by specific applications involving nonconservative vector fields, namely the secondary bifurcations from nontrivial equilibria in hydrodynamical systems such as Couette–Taylor flows and Rayleigh–Bénard convection in a spherical shell. The

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problem was attacked analytically by Chossat and Iooss [7], and more qualitatively by Rand [37]. A major success of the analytical approach was obtained by Iooss [19], who classified the possible patterns bifurcating from a group orbit of equilibria in a system with symmetry $O(2)$. In Moutrane [32], the bifurcation of rotating waves, which are relative equilibria with a single drift frequency, was investigated in the problem of the onset of convection in a system with spherical symmetry. However it was Krupa [20] who first developed a general theory for bifurcations from relative equilibria. The basic tool he used was the *Invariant Theorem* of Palais (see [4,36]). If G is a Lie group acting properly on a manifold M, the Slice Theorem establishes an isomorphism for each $m \in M$ between a tubular neighborhood of the orbit $G \cdot m$ and the *normal bundle*, with base $G \cdot m$ and fiber equal to the normal slice N_m to the tangent space to $G \cdot m$ at m. It was shown by Field [11] and then by Krupa that within such a tubular neighborhood any G-equivariant vector field $X \in \mathfrak{X}(M)$ can be decomposed into the sum of two vector fields: one, X_N, defined on the normal bundle, and the other, X_T, defined on the tangent bundle to $G \cdot m$. Krupa showed that the dynamical information, in particular the bifurcation properties for a parameter dependent family of vector fields, are entirely contained in X_N.

The analysis of relative equilibria of conservative systems has played a central role in the development of geometric mechanics, ranging from the classic work of Riemann [38] and Routh [40,41] to Smale's seminal work [43]. However, the use of local singularity theory methods, rather than explicit calculations or global topological methods, in the analysis of conservative systems is relatively recent. (See, e.g., [10,14,18,22,29] and the references discussed below.) Bifurcations of relative equilibria of Lagrangian systems and canonical Hamiltonian systems, i.e., Hamiltonian systems on cotangent bundles, with the canonical symplectic structure and a lifted group action, have been studied by Lewis et al. [27] and Lewis [24,25] using the reduced energy-momentum method developed in [42] and [23]. This approach uses the locked Lagrangian, the generalization of Smale's augmented potential to Lagrangian systems and their Hamiltonian analogs, to characterize relative equilibria as critical points of functions on the configuration manifold parameterized by elements of the algebra $\mathfrak{g}$ of the symmetry group G. A key component of the reduced energy-momentum method is the decomposition of the tangent space $T_q Q$ of the configuration manifold Q at a point q into the tangent space $\mathfrak{g} \cdot q$ to the group orbit and an appropriate complement consisting of so-called 'internal' variations. The associated decomposition of the relative equilibrium equations into 'rigid' and 'internal' equilibrium conditions is analogous to the decompositions introduced by Field [11] and Krupa [20] in the context of general equivariant vector fields. The 'rigid' condition can be used to determine a submanifold of 'candidate relative equilibria;' imposing the remaining equilibrium conditions on this submanifold determines the relative equilibria.

Our goal is the development in the symplectic category of a decomposition tool analogous to that of Krupa that will take into account the additional structure present at the kinematical level in Hamiltonian systems, without assuming all the ingredients utilized in the reduced energy-momentum method. Given that many Hamiltonian systems are constructed on symplectic manifolds that are not cotangent bundles, such a tool is of much interest. The analog of the Invariant Slice Theorem in the symplectic category is given by the *Marle–Guillemin–Sternberg normal form* [16,17,28] (we will refer to it as the *MGS–normal form*) so, in principle, one could work as in Krupa [20] using this normal form instead of the Slice Theorem. This does not seem to be the best way to proceed, since to search for relative equilibria of Hamiltonian systems one does not need to work with the Hamiltonian vector field; there are scalar functions, the *augmented Hamiltonians*, whose critical points are precisely the relative equilibria. Guided by Krupa's normal bundle decomposition for equivariant vector fields and the MGS-normal form, in Section 2 we will construct a *slice mapping* with which we can decompose the critical point equations determining the relative equilibria into a system of four equations. These split critical point equations are analyzed in Section 3 in a neighborhood of a given a relative equilibrium m_e. Using the Implicit Function Theorem and Lyapunov–Schmidt reduction, we can construct a local submanifold containing all relative equilibria sufficiently near the group orbit of m_e. The remaining equilibrium conditions, called the *reduced critical point equations*, can be analyzed on this submanifold using standard techniques from bifurcation theory. In Section 3.1 we study the equivariance properties of the reduced critical point equations. In Section 3.2 we show how to choose a slice mapping so that one of the reduced critical point equations admits a simpler solution.

In Section 4 we use the reduced critical equations and a slice mapping constructed via the MGS-normal form to study the persistence of a family of relative equilibria in a neighborhood of a nondegenerate relative equilibrium when the symmetry group of the system is Abelian. In particular, we generalize to proper group actions a result from Lerman and Singer [21] originally proven for compact groups. This result was already presented in [33].

In Section 5 we study bifurcations from a degenerate relative equilibrium and find Hamiltonian analogs to bifurcation theorems for solutions with maximal isotropy which were first stated in the nonconservative context, namely the Equivariant Branching Lemma of Vanderbauwhede [44] and Cicogna [9], and a theorem for bifurcation of solutions with maximal isotropy group of complex type [8,30].

In Section 6 we apply the results developed here to a system on $\mathbb{C}^4$ modeling a $1:2$ wave resonance. Such models have been analyzed in [5] and [6]; thus this example allows a comparison of our approach to previously employed techniques.

2. Relative equilibria as critical points

Let G be a Lie group acting smoothly on the manifold M and let $X \in \mathfrak{X}(M)$ be a smooth G-equivariant vector field on M with flow F_t. We now briefly introduce some of the key notations used here. Let $\exp : \mathfrak{g} \to G$ denote the exponential map from the Lie algebra $\mathfrak{g}$ of G to G, $g \cdot m$ denote the image of $m \in M$ under the action of $g \in G$, and ξ_M denote the vector field

$$\xi_M(m) = \frac{\mathrm{d}}{\mathrm{d}\epsilon} \exp(\epsilon\xi) \cdot m|_{\epsilon=0},$$

called the *infinitesimal generator* associated to $\xi \in \mathfrak{g}$. Given a subspace $\mathfrak{s}$ of $\mathfrak{g}$ and a point $m \in M$, we set

$$\mathfrak{s} \cdot m := \left\{ \xi_M(m) \;\middle|\; \xi \in \mathfrak{s} \right\} \subset T_m(G \cdot m),$$

where $G \cdot m := \{g \cdot m \mid g \in G\}$ denotes the orbit of m. If M and N are manifolds and $\varphi : M \to N$ is a differentiable map, then the linearization of φ at $m \in M$ is denoted by $T_m\varphi : T_m M \to T_{\varphi(m)} N$; if N is a vector space, then the linear map $D\varphi(m) : T_m M \to N$ is defined using the standard identification of $T_{\varphi(m)} N$ with N. If F maps a product $V_1 \times \cdots \times V_k$ of vector spaces into a vector space N, then $D_{V_j} F(v_1, \ldots, v_k) : V_j \to N$ denotes the partial derivative of F with respect to the jth factor. Given a subspace W of a vector space V, $W^\circ \subset V^*$ denotes the annihilator of W.

Let $X \in \mathfrak{X}(M)$ be a smooth G-equivariant vector field on M with flow F_t. We say that the point $m_e \in M$ is a *relative equilibrium* of the vector field X if there exists an element ξ of the Lie algebra $\mathfrak{g}$ of G, called a *generator of the relative equilibrium*, such that $F_t(m_e) = \exp(t\xi) \cdot m_e$; m_e is a relative equilibrium with generator ξ if and only if $X(m_e) = \xi_M(m_e)$.

We are interested in relative equilibria of Hamiltonian systems. Specifically, we assume throughout the paper that the manifold M is symplectic and that the action of G on M is symplectic, with associated equivariant momentum map $\mathbf{J} : M \to \mathfrak{g}^*$. In addition, we assume that the equivariant vector field X is Hamiltonian, with associated G-invariant Hamiltonian $h \in C^\infty(M)$. In this framework, the search for relative equilibria reduces to the determination of the critical points of a certain family of functions. Indeed, a classical result ([1, p. 307] and [3, p. 380]) states that a point $m_e \in M$ is a relative equilibrium of X_h with generator $\xi \in \mathfrak{g}$ if and only if m_e is a critical point of the *augmented Hamiltonian* h^ξ, given by

$$h^\xi(m) := h(m) - \langle \mathbf{J}(m), \xi \rangle$$

for all $m \in M$. Thus, our algorithm is intended to identify the pairs $(m_e, \xi) \in M \times \mathfrak{g}$ such that

$$Dh^\xi(m_e) = 0. \tag{1}$$

Note that if m_e has nontrivial continuous symmetry, i.e. $\mathfrak{g}_{m_e} = \{\zeta \in \mathfrak{g} \mid \zeta_M(m_e) = 0\} \neq \{0\}$, then the generator of m_e is not unique. If ξ is a generator of a relative equilibrium m_e, then for any $\zeta \in \mathfrak{g}_{m_e}$, $\xi + \zeta$ is also a generator.

The main goal of this section is the decomposition of the *relative equilibrium equation* (1) into a systems of four equations, each defined on a space determined by the geometry of the problem.

Assume that m_e is a relative equilibrium with generator ξ and momentum $\mu := \mathbf{J}(m_e)$. Let $\mathfrak{g}_{m_e}$ denote the Lie algebra of the isotropy subgroup G_{m_e} of m_e and $\mathfrak{g}_\mu$ the Lie algebra of the isotropy subgroup G_μ of μ. Choose complements $\mathfrak{q}$ of $\mathfrak{g}_\mu$ in $\mathfrak{g}$ and $\mathfrak{m}$ of $\mathfrak{g}_{m_e}$ in $\mathfrak{g}_\mu$, so that

$$\mathfrak{g} = \mathfrak{g}_\mu \oplus \mathfrak{q} = \mathfrak{g}_{m_e} \oplus \mathfrak{m} \oplus \mathfrak{q}. \tag{2}$$

The symbols i and $\mathbb{P}$ with appropriate subscripts will denote the natural injections and projections determined by the splittings (2). For instance $i_{\mathfrak{g}_{m_e}} : \mathfrak{g}_{m_e} \to \mathfrak{g} = \mathfrak{g}_{m_e} \oplus \mathfrak{m} \oplus \mathfrak{q}$ is the canonical injection of $\mathfrak{g}_{m_e}$ into $\mathfrak{g}$ and $\mathbb{P}_{\mathfrak{g}_{m_e}} : \mathfrak{g} = \mathfrak{g}_{m_e} \oplus \mathfrak{m} \oplus \mathfrak{q} \to \mathfrak{g}_{m_e}$ extracts the $\mathfrak{g}_{m_e}$ component of any vector in $\mathfrak{g}$.

Definition 2.1. Let V be a vector space and $\mathcal{U} \subset \mathfrak{m}^* \times V$ be an open neighborhood of $(0,0) \in \mathfrak{m}^* \times V$. A smooth mapping $\Psi : \mathcal{U} \subset \mathfrak{m}^* \times V \to M$ is said to be a *slice mapping* at the point $m_e \in M$ if Ψ is a diffeomorphism onto its image satisfying the following conditions:

(SM1) $\Psi(0,0) = m_e$.
(SM2) For any $(\eta, v) \in \mathcal{U}$

$$T_{\Psi(\eta,v)}M = (\mathfrak{m} \oplus \mathfrak{q}) \cdot \Psi(\eta, v) + T_{(\eta,v)}\Psi(\mathfrak{m}^* \times V). \tag{3}$$

(SM3) The pullback $\mathbf{j} := \mathbf{J} \circ \Psi : \mathcal{U} \to \mathfrak{g}^*$ of the momentum map satisfies

$$D\mathbf{j}(0,0)(\delta\eta, \delta v) = \mathbb{P}^*_{\mathfrak{m}}\delta\eta \tag{4}$$

for all $\delta\eta \in \mathfrak{m}^*$ and $\delta v \in V$.

In the following proposition we show that given a coordinate chart at a point m in a finite-dimensional manifold M, we can explicitly construct a slice mapping Ψ at m.

Proposition 2.2. *Let $\psi : U \subset X \to M$ be a coordinate chart at a point m in a finite-dimensional manifold M and let V and W be subspaces of the vector space X such that*

(i) $\psi(0) = m$,
(ii) $T_0\psi(V)$ *is a complement to* $\mathfrak{m} \cdot m$ *in* $\ker D\mathbf{J}(m)$,
(iii) *the map*

$$\begin{aligned} A : W &\to (\mathfrak{g}_m \oplus \mathfrak{q})^\circ \\ w &\mapsto D\mathbf{J}(m)(T_0\psi w), \end{aligned}$$

is an isomorphism.

Let V' and W' be neighborhoods of the origin in V and W such that $V' \times W' \subset U$ and set $\mathcal{U} := i^*_{\mathfrak{m}}(AW') \times V' \subset \mathfrak{m}^* \times V$. Then the map

$$\begin{aligned} \Psi : \mathcal{U} \subset \mathfrak{m}^* \times V &\to M \\ (\eta, v) &\mapsto \psi\big(v + A^{-1}\mathbb{P}^*_{\mathfrak{m}}\eta\big) \end{aligned}$$

is a slice mapping at $m \in M$.

Proof. Property (SM1) follows trivially from (i). Property (SM3) follows from (ii), (iii), and the definition of Ψ.

As the first step in the proof of (SM2), we show that (3) holds at $(0,0)$. Note that (SM3) implies that

$$\ker D\mathbf{J}(m) \cap T_{(0,0)}\Psi\big(\mathfrak{m}^* \times \{\mathbf{0}\}\big) = \{\mathbf{0}\}. \tag{5}$$

Combining (ii), (2), and (5), we obtain

$$\begin{aligned} \dim T_{(0,0)}\Psi(\mathfrak{m}^* \times V) &= \dim \mathfrak{m} + \dim V = \dim\big(\ker D\mathbf{J}(m)\big) \\ &= \dim M - \dim(\mathfrak{g} \cdot m) = \dim M - \dim \mathfrak{m} - \dim \mathfrak{q}. \end{aligned} \tag{6}$$

If $\zeta_M(m) = T_{(0,0)}\Psi(\delta\eta, \delta v)$, then, since Ψ satisfies (SM3),

$$D\mathbf{J}(m)\zeta_M(m) = D\mathbf{J}\big(\Psi(0,0)\big)\big(T_{(0,0)}\Psi(\delta\eta, \delta v)\big) = \mathbb{P}^*_{\mathfrak{m}}\delta\eta.$$

On the other hand, equivariance of $\mathbf{J}$ implies that

$$D\mathbf{J}(m)\zeta_M(m) = -\operatorname{ad}^*_\zeta \mu \in \mathfrak{m}^\circ.$$

Hence $\operatorname{ad}^*_\zeta \mu = 0$, i.e., $\zeta \in \mathfrak{g}_\mu$, and $\zeta_M(m) \in \mathfrak{g}_\mu \cdot m = \mathfrak{m} \cdot m$. Thus condition (ii) implies that $\zeta_M(m) = 0$. Combining this result with (2) and (6) shows that (3) is valid at $(0,0)$.

We now show that (3) holds for any $(\eta, v) \in \mathcal{U}$. Let $\{\xi_1, \dots, \xi_j\}$, $\{\eta_1, \dots, \eta_k\}$, and $\{v_1, \dots, v_\ell\}$ be bases for $\mathfrak{m} \oplus \mathfrak{q}$, $\mathfrak{m}^*$, and V. Define the maps $u_i : \mathcal{U} \to TM$, $i = 1, \dots, j + k + \ell$, by

$$u_i(\eta, v) := \begin{cases} (\xi_i)_M\big(\Psi(\eta, v)\big), & 1 \leqslant i \leqslant j, \\ T_{(\eta,v)}\Psi(\eta_{i-j}, 0), & j < i \leqslant j + k, \\ T_{(\eta,v)}\Psi(0, v_{i-j-k}), & j + k < i \leqslant j + k + \ell. \end{cases}$$

The arguments given above show that $\{u_1(0,0), \dots, u_{j+k+\ell}(0,0)\}$ is a basis for $T_{m_e}M$. Since linear independence is an open condition, $\{u_1(\eta, v), \dots, u_{j+k+\ell}(\eta, v)\}$ is a basis of $T_{\Psi(\eta,v)}M$ for (η, v) sufficiently near the origin. In particular,

$$\begin{aligned} T_{\Psi(\eta,v)}M &= \operatorname{span}\big\{u_1(\eta, v), \dots, u_{j+k+\ell}(\eta, v)\big\} \\ &= \operatorname{span}\big\{(\xi_1)_M\big(\Psi(\eta, v)\big), \dots, (\xi_j)_M\big(\Psi(\eta, v)\big)\big\} \\ &\quad \oplus \operatorname{span}\big\{T_{(\eta,v)}\Psi(\eta_1, 0), \dots, T_{(\eta,v)}\Psi(\eta_k, 0)\big\} \\ &\quad \oplus \operatorname{span}\big\{T_{(\eta,v)}\Psi(0, v_1), \dots, T_{(\eta,v)}\Psi(0, v_\ell)\big\} \\ &= (\mathfrak{m} \oplus \mathfrak{q}) \cdot \Psi(\eta, v) \oplus T_{(\eta,v)}\Psi(\mathfrak{m}^* \times V), \end{aligned}$$

as required. □

The introduction of a slice mapping Ψ allows us to decompose the critical point equation (1) into a system of four equations. Using property (SM2) of the slice mapping and setting $\mathcal{H}^{\xi} := h^{\xi} \circ \Psi$, we see that the point $\Psi(\eta, v) \in M$ is a relative equilibrium with generator ξ if and only if

$$\begin{cases} \text{(RE1)} & i_{\mathfrak{q}}^{*} \operatorname{ad}_{\xi}^{*} \mathbf{j}(\eta, v) = 0, \\ \text{(RE2)} & i_{\mathfrak{m}}^{*} \operatorname{ad}_{\xi}^{*} \mathbf{j}(\eta, v) = 0, \\ \text{(RE3)} & D_{\mathfrak{m}^{*}} \mathcal{H}^{\xi}(\eta, v) = 0, \\ \text{(RE4)} & D_{V} \mathcal{H}^{\xi}(\eta, v) = 0. \end{cases} \tag{7}$$

Remark 2.3. If symmetry is broken in a neighborhood of m_e, then $\mathfrak{g}_{m_e} \cdot \Psi(\eta, v)$ is typically nontrivial. In this case, the first two conditions alone do not guarantee that the rigid condition $\operatorname{ad}_{\xi}^{*} \mathbf{j}(\eta, v) = 0$ is satisfied. However, if (RE3) and (RE4) are satisfied, then $D\mathcal{H}^{\xi}(\eta, v) = 0$; in particular, if $\zeta \in \mathfrak{g}_{m_e}$, then (SM2) implies that there exist $\delta\eta \in \mathfrak{m}^{*}$ and $\delta v \in V$ such that $\zeta_M(\Psi(\eta, v)) = T_{(\eta, v)}\Psi(\delta\eta, \delta v)$ and hence

$$\begin{aligned} \left\langle \operatorname{ad}_{\xi}^{*} \mathbf{j}(\eta, v), \zeta \right\rangle &= \left\langle \operatorname{ad}_{\xi}^{*} \mathbf{J}\big(\Psi(\eta, v)\big), \zeta \right\rangle = -Dh^{\xi}\big(\Psi(\eta, v)\big)\zeta_M\big(\Psi(\eta, v)\big) \\ &= -D\mathcal{H}^{\xi}(\eta, v)(\delta\eta, \delta v) = 0. \end{aligned}$$

Combining this with (RE1) and (RE2) yields the rigid equilibrium condition $\operatorname{ad}_{\xi}^{*} \mathbf{j}(\eta, v) = 0$.

Remark 2.4. Note that in order to split the critical point equation (1) into (7), only property (SM2) of the slice mapping was utilized. As we shall see in the following section, property (SM3) simplifies the analysis of Eqs. (7). Equations (RE1) and (RE3) are, by construction, nondegenerate in the sense that implicit solutions to these equations always exist. Thus the bifurcation analysis is carried out only on the equations obtained by substituting the solutions of (RE1) and (RE3) into (RE2) and (RE4).

3. The reduced critical point equations

In this section we start with a relative equilibrium m_e with generator $\xi \in \mathfrak{g}$ and derive a minimal set of mappings and equations, called the *reduced critical point equations*, determining the relative equilibria in a neighborhood of m_e. We proceed in three steps, using the Implicit Function Theorem and Lyapunov–Schmidt reduction (see for instance [13]). In each step we indicate sufficient technical hypotheses to guarantee that the step can be carried out for infinite-dimensional systems. We emphasize that the construction of the reduced equations is not an 'all or nothing' procedure; if some of the hypotheses are not satisfied, the relevant steps can be modified or omitted, yielding analogous, although possibly less convenient, bifurcation equations.

Step 1. Using the notation introduced in Definition 2.1, let $F_1 : \mathcal{U} \times \mathfrak{g}_{m_e} \times \mathfrak{m} \times \mathfrak{q} \to \mathfrak{q}^{*}$ be the mapping given by

$$F_1(\eta, v, \alpha, \beta, \gamma) := i_{\mathfrak{q}}^{*} \operatorname{ad}_{\alpha+\beta+\gamma}^{*} \mathbf{j}(\eta, v),$$

with differential

$$\begin{aligned} DF_1(\mathbf{0})(\delta\eta, \delta v, \delta\alpha, \delta\beta, \delta\gamma) &= i_{\mathfrak{q}}^*\big(\mathrm{ad}^*_{\delta\alpha+\delta\beta+\delta\gamma}\,\mathbf{j}(0,0) + \mathrm{ad}^*_\xi\big(D\mathbf{j}(0,0)(\delta\eta, \delta v)\big)\big) \\ &= i_{\mathfrak{q}}^*\big(\mathrm{ad}^*_{\delta\gamma}\,\mu + \mathrm{ad}^*_\xi\big(\mathbb{P}^*_{\mathfrak{m}}\delta\eta\big)\big). \end{aligned}$$

Here we used property (SM3) of the slice mapping Ψ.

Since $\delta\gamma \mapsto i_{\mathfrak{q}}^*(\mathrm{ad}^*_{\delta\gamma}\,\mu)$ is an isomorphism between $\mathfrak{q}$ and $\mathfrak{q}^*$, we conclude that the partial derivative $D_{\mathfrak{q}}F_1(\mathbf{0})$ is an isomorphism. Thus the Implicit Function Theorem implies that there is a function $\gamma : \mathcal{U}_1 \subset \mathcal{U} \times \mathfrak{g}_{m_e} \times \mathfrak{m} \to \mathfrak{q}$ such that

$$F_1\big(\eta, v, \alpha, \beta, \gamma(\eta, v, \alpha, \beta)\big) = i_{\mathfrak{q}}^*\,\mathrm{ad}^*_{\xi+\alpha+\beta+\gamma(\eta,v,\alpha,\beta)}\,\mathbf{j}(\eta, v) = 0$$

for all $(\eta, v, \alpha, \beta) \in \mathcal{U}_1$. In other words, we have found a $\mathfrak{m}^* \times V \times \mathfrak{g}_{m_e} \times \mathfrak{m}$-parameter family of points that satisfy part (RE1) of the split critical point equations. Set

$$\omega_1(\eta, v, \alpha, \beta) := \xi + \alpha + \beta + \gamma(\eta, v, \alpha, \beta). \tag{8}$$

Step 2. In this step we assume that the subspace $\mathfrak{m}$ is reflexive, that is, $\mathfrak{m}^{**} \simeq \mathfrak{m}$. (Since $\dim\mathfrak{m} \leqslant \dim M$, this hypothesis is nontrivial only if both M and G are infinite-dimensional.) We now construct a $\mathfrak{m}^* \times V \times \mathfrak{g}_{m_e}$-parameter family of points satisfying the relative equilibrium equations (RE1) and (RE3) by applying the Implicit Function Theorem to (RE3), solving for the $\mathfrak{m}$ component of the family of points constructed in Step 1.

Let $F_2 : \mathcal{U}_1 \subset \mathfrak{m}^* \times V \times \mathfrak{g}_{m_e} \times \mathfrak{m} \to \mathfrak{m}^{**} \simeq \mathfrak{m}$ be the mapping defined by $F_2(\eta, v, \alpha, \beta) := D_{\mathfrak{m}^*}\mathcal{F}(\eta, v, \omega_1(\eta, v, \alpha, \beta))$. Since we intend to solve the equation $F_2 = 0$ for the $\mathfrak{m}$ parameter using the Implicit Function Theorem, we compute $D_{\mathfrak{m}}F_2(0, 0, 0, 0)$. Given arbitrary $\delta\beta \in \mathfrak{m}$ and $\delta\eta \in \mathfrak{m}^*$,

$$\begin{aligned} \big\langle \delta\eta, D_{\mathfrak{m}}F_2(\mathbf{0})\delta\beta\big\rangle &= \frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{H}^{\omega_1(0,0,0,t\delta\beta)}(s\delta\eta, 0)\Big|_{t=0}\Big|_{s=0} \\ &= \big\langle D\mathbf{j}(0,0)(\delta\eta, 0), D_{\mathfrak{m}}\omega_1(\mathbf{0})\delta\beta\big\rangle = \big\langle \mathbb{P}^*_{\mathfrak{m}}\delta\eta, D_{\mathfrak{m}}\omega_1(\mathbf{0})\delta\beta\big\rangle \\ &= \big\langle \delta\eta, \mathbb{P}_{\mathfrak{m}}(\delta\beta + D_{\mathfrak{m}}\gamma(\mathbf{0})\delta\beta)\big\rangle = \langle \delta\eta, \delta\beta\rangle \end{aligned}$$

follows from property (SM3) of the slice map and the formula (8) for the generator ω_1. Hence $D_{\mathfrak{m}}F_2(\mathbf{0})$ is the identity map. The Implicit Function Theorem thus implies that there is a function $\beta : \mathcal{U}_2 \subset \mathcal{U} \times \mathfrak{g}_{m_e} \to \mathfrak{m}$ satisfying $F_2(\eta, v, \alpha, \beta(\eta, v, \alpha)) = D_{\mathfrak{m}^*}\mathcal{F}(\eta, v, \omega_1(\eta, v, \alpha, \beta(\eta, v, \alpha))) = 0$ for all $(\eta, v, \alpha) \in \mathcal{U}_2$. Set

$$\omega_2(\eta, v, \alpha) := \omega_1\big(\eta, v, \alpha, \beta(\eta, v, \alpha)\big).$$

Step 3. We now treat the (RE4) component of the relative equilibrium equation. We use the standard Lyapunov–Schmidt reduction procedure of bifurcation theory to partially solve (RE4).

Let $\mathcal{L}: V \to V^*$ denote the linear transformation satisfying

$$\langle \mathcal{L}v, w\rangle := D_{VV}\mathcal{H}^{\xi}(0,0)(v,w) = D^2\mathcal{H}^{\xi}(0,0)\big((0,v),(0,w)\big)$$

for all v and $w \in V$. Set $V_0 := \ker \mathcal{L}$ and choose closed subspaces $V_1 \subset V$ and $V_2 \subset V^*$ such that

$$V = V_0 \oplus V_1 \quad \text{and} \quad V^* = \operatorname{range} \mathcal{L} \oplus V_2. \tag{9}$$

If V is infinite dimensional, additional hypotheses are needed to guarantee the existence of closed complements V_1 and V_2. For example, it suffices that V is a Banach space and $\mathcal{L}$ is a Fredholm operator.

Let $\mathbb{P}: V^* \to V_2$ denote the projection determined by the decomposition (9) of V^*. Define $F_3: \mathfrak{m}^* \times V_0 \times V_1 \times \mathfrak{g}_{m_e} \to \operatorname{range} \mathcal{L}$ by

$$F_3(\eta, v_0, v_1, \alpha) := (\mathbb{I} - \mathbb{P}) D_V \mathcal{H}^{\omega_2(\eta, v_0+v_1, \alpha)}(\eta, v_0 + v_1).$$

Using the Implicit Function Theorem once more, we can solve the equation $F_3(\eta, v_0, v_1, \alpha) = 0$ for v_1. The identities $(\mathbb{I} - \mathbb{P})\mathcal{L} = \mathcal{L}$ and $D_V \mathbf{j}(0,0)$ imply that $D_{V_1} F_3(\mathbf{0}) = \mathcal{L}|_{V_1}$. $\mathcal{L}|_{V_1}$ is, by construction, an isomorphism of V_1 onto $\operatorname{range} \mathcal{L}$ and the Implicit Function Theorem guarantees the existence of a neighborhood $\mathcal{U}_3$ of $(0,0,0) \in \mathfrak{m}^* \times V_0 \times \mathfrak{g}_{m_e}$ and a local function $v_1: \mathcal{U}_3 \to V_1$ such that

$$F_3\big(\eta, v_0, v_1(\eta, v_0, \alpha), \alpha\big) = 0,$$

for any $(\eta, v_0, \alpha) \in \mathcal{U}_3$.

Define the *generator map* $\Xi: \mathcal{U}_3 \to \mathfrak{g}$, $\rho: \mathcal{U}_3 \to \mathfrak{m}^*$, and $B: \mathcal{U}_3 \to V_2$ by

$$\begin{aligned} \Xi(\eta, v_0, \alpha) &:= \omega_2\big(\eta, v_0 + v_1(\eta, v_0, \alpha), \alpha\big), \\ \rho(\eta, v_0, \alpha) &:= \iota_{\mathfrak{m}}^* \operatorname{ad}^*_{\Xi(\eta, v_0, \alpha))} \mathbf{j}\big((\eta, v_0 + v_1(\eta, v_0, \alpha)\big), \\ B(\eta, v_0, \alpha) &:= \mathbb{P} D_V \mathcal{H}^{\Xi(\eta, v_0, \alpha)}\big(\eta, v_0 + v_1(\eta, v_0, \alpha)\big). \end{aligned}$$

In a sufficiently small neighborhood $\mathcal{U}_3$ of the origin any solution (η, v_0, α) of the equations

$$\begin{cases} \text{(B1)} \quad \rho(\eta, v_0, \alpha) = 0, \\ \text{(B2)} \quad B(\eta, v_0, \alpha) = 0 \end{cases} \tag{10}$$

determines a relative equilibrium $\Psi(\eta, v_0 + v_1(\eta, v_0, \alpha))$ with generator $\Xi(\eta, v_0, \alpha)$. On the other hand, any relative equilibrium m sufficiently near m_e in the slice $\Psi(\mathfrak{m}^* \times V)$ satisfies $m = \Psi(\eta, v_0 + v_1(\eta, v_0, \alpha))$ for some solution (η, v_0, α) of (B1) and (B2); any generator ξ of m satisfies $\xi - \Xi(\eta, v_0, \alpha) \in \mathfrak{g}_m$. Equations (B1) and (B2) will be usually referred to as the *rigid residual equation* and the *bifurcation equation*, respectively. Let

$R : \mathfrak{m}^* \times V_0 \times \mathfrak{g}_{m_e} \to \mathfrak{m}^* \times V_2$ be the product of ρ and B, that is,

$$\begin{aligned} R : \mathfrak{m}^* \times V_0 \times \mathfrak{g}_{m_e} &\to \mathfrak{m}^* \times V_0 \\ (\eta, v_0, \alpha) &\mapsto \big(\rho(\eta, v_0, \alpha), B(\eta, v_0, \alpha)\big). \end{aligned}$$

We will refer to the equality

$$R(\eta, v_0, \alpha) = 0 \tag{11}$$

as the *reduced critical point equations*.

Note that since the operator $\mathcal{L}$ satisfies $\langle \mathcal{L}v, w\rangle = \langle \mathcal{L}w, v\rangle$ for all v and $w \in V$, the spaces V_0 and V_2 can be naturally identified using the inner product when V is a Hilbert space.

Remark 3.1. Note that even though the critical point equations (1) determining the relative equilibria in our situation can be naturally understood as a gradient equation when M is a Riemannian Hilbert manifold, this analytic feature is not in general available for the reduced version (11) of these equations.

The gradient character of (1) *is* preserved by the reduction procedure when the relative equilibrium m_e is a true equilibrium with total isotropy. In this case, $\mathfrak{g}_{m_e} = \mathfrak{g}$; thus $\mathfrak{m} = \mathfrak{q} = \{0\}$ and the rigid residual equation (B1) is trivial. As we will now show, if $X(m_e) = 0$, $G_{m_e} = G$, and V is a Hilbert space, then the bifurcation equation (B2) is a gradient equation. Our analysis very closely follows the one introduced in [12].

If $\mathfrak{m} = \mathfrak{q} = \{0\}$, then any coordinate chart $\psi : \mathcal{U} \subset X \to M$ such that $\psi(0) = m_e$ is a slice mapping at m_e, with $V = X$, and the critical point equations (RE1)–(RE4) collapse to the single equation $D\mathcal{H}^\xi(v) = 0$. In this situation only the third step of the general procedure, the Lyapunov–Schmidt reduction, is nontrivial.

We fix an inner product $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ on V and denote by $\nabla \mathcal{H}^\xi(v)$ the usual gradient of $\mathcal{H}^\xi$ with respect to $\langle\!\langle \cdot, \cdot \rangle\!\rangle$, i.e.,

$$\big\langle\!\big\langle \nabla \mathcal{H}^\xi(v), w \big\rangle\!\big\rangle = D\mathcal{H}^\xi(v)w$$

for any $w \in V$. If m_e is a relative equilibrium with generator ξ, the relative equilibria near m_e correspond to the zeroes of the map $F : V \times \mathfrak{g} \to V$ defined by

$$F(v, \alpha) = \nabla \mathcal{H}^{\xi+\alpha}(v).$$

Let $L : V \to V$ be the mapping defined by $L(v) = D_V F(0, 0)v$. It can easily be verified that

$$\big\langle\!\big\langle L(v), w \big\rangle\!\big\rangle = D^2\mathcal{H}^\xi(0)(v, w)$$

for any v and $w \in V$. Note that the mapping L is a self-adjoint operator; hence if we set $V_0 = \ker L$ and $V_1 = \operatorname{range} L$, then V has the orthogonal decomposition $V = V_0 \oplus V_1$. Let $\mathbb{P} : V \to V_0$ denote the canonical projection with respect to the splitting $V = V_0 \oplus V_1$. Now, if we decompose $v \in V$ as $v = v_0 + v_1$, with $v_0 \in V_0$ and $v_1 \in V_1$, and apply the Implicit Function Theorem to the equation

$$(\mathbb{I} - \mathbb{P})F(v_0 + v_1, \alpha) = 0,$$

we obtain a function $v_1 : V_0 \times \mathfrak{g} \to V_1$ such that

$$(\mathbb{I} - \mathbb{P})F\big(v_0 + v_1(v_0, \alpha), \alpha\big) = 0. \tag{12}$$

Thus, in this case, the bifurcation equation is

$$B(v_0, \alpha) = \mathbb{P}F\big(v_0 + v_1(v_0, \alpha), \alpha\big) = 0.$$

We now show that the map B is the gradient of $g(v_0, \alpha) := \mathcal{H}^{\xi+\alpha}(v_0 + v_1(v_0, \alpha))$; that is,

$$B(v_0, \alpha) = \nabla_{V_0} g(v_0, \alpha).$$

Indeed, note that for any $w \in V_0$

$$\begin{aligned} \big\langle\!\big\langle \nabla_{V_0} g(v_0, \alpha), w \big\rangle\!\big\rangle &= D\mathcal{H}^{\xi+\alpha}\big(v_0 + v_1(v_0, \alpha)\big)\big(w + D_{V_0} v_1(v_0, \alpha) w\big) \\ &= \big\langle\!\big\langle F\big(v_0 + v_1(v_0, \alpha), \alpha\big), \mathbb{P}w + (\mathbb{I} - \mathbb{P}) D_{V_0} v_1(v_0, \alpha) w \big\rangle\!\big\rangle \\ &= \big\langle\!\big\langle \mathbb{P}F\big(v_0 + v_1(v_0, \alpha), \alpha\big), w \big\rangle\!\big\rangle = \big\langle\!\big\langle B(v_0, \alpha), w \big\rangle\!\big\rangle, \end{aligned}$$

since $w \in V_0 = \operatorname{range} \mathbb{P}$, $D_{V_0} v_1(v_0, \alpha) w \in V_1 = \operatorname{range}(\mathbb{I} - \mathbb{P})$, $\mathbb{P}$ is self-adjoint, and (12) is satisfied.

3.1. The equivariance properties of the reduced critical point equations

The symmetries of the relevant equations play an important role the solution of a bifurcation problem (see, for instance, [15]). We will see that if the G-action on M is *proper*, then the relative equilibrium equations (B1) and (B2) can be constructed so as to be equivariant with respect to the induced action of $G_{m_e,\xi} := G_{m_e} \cap G_\xi$ on $\mathfrak{m}^* \times V_0$. Here G_ξ denotes the isotropy subgroup of the generator $\xi \in \mathfrak{g}$ of the relative equilibrium $m_e \in M$ with respect to the adjoint action of G on $\mathfrak{g}$.

An *equivariant slice mapping* is a slice mapping $\Psi : \mathcal{U} \subset \mathfrak{m}^* \times V \to M$ satisfying the additional condition

(ESM) The subspace $\mathfrak{m}^*$ of $\mathfrak{g}^*$ is $\mathrm{Ad}^*_{G_{m_e,\xi}}$-invariant and the slice mapping $\Psi : \mathcal{U} \subset \mathfrak{m}^* \times V \to M$ is $G_{m_e,\xi}$-equivariant with respect to the coadjoint action of $G_{m_e,\xi}$ on $\mathfrak{m}^*$ and some action of $G_{m_e,\xi}$ on V.

Note that since the group $G_{m_e,\xi}$ is compact and fixes $(0, 0) \in \mathfrak{m}^* \times V$, the open neighborhood $\mathcal{U}$ of $(0, 0) \in \mathfrak{m}^* \times V$ in (ESM) can always be chosen to be $G_{m_e,\xi}$-invariant.

Proposition 3.2. *If the group G acts properly on M and the coordinate chart $\psi : \mathcal{U} \subset X \to M$ with $\psi(0) = m_e$ is equivariant with respect to some action of $G_{m_e,\xi}$ on X, then the subspaces $\mathfrak{m}$, $\mathfrak{q}$, V, and W can be taken to be $G_{m_e,\xi}$ invariant. For these choices, the slice mapping constructed in Proposition* 2.2 *is $G_{m_e,\xi}$-equivariant.*

Proof. First we show that $G_{m_e,\xi}$-invariant decompositions $\mathfrak{g} = \mathfrak{g}_{m_e} \oplus \mathfrak{m} \oplus \mathfrak{q}$ and $X = V \oplus W$ exist. Note that the isotropy subgroup G_{m_e} is compact, since the action of G on M is assumed to be proper; consequently the subgroup $G_{m_e,\xi}$ is also compact. This guarantees the existence of a $\mathrm{Ad}_{G_{m_e,\xi}}$-invariant inner product on $\mathfrak{g}$, which we can use to determine a $\mathrm{Ad}_{G_{m_e,\xi}}$-invariant decomposition $\mathfrak{g} = \mathfrak{g}_{m_e} \oplus \mathfrak{m} \oplus \mathfrak{q}$ of the Lie algebra.

The orthogonal complement of $\mathfrak{g}_\mu \cdot m_e$ in $\ker D\mathbf{J}(m_e)$ with respect to a $G_{m_e,\xi}$-invariant inner product is an invariant subspace. Hence the pre image with respect to the equivariant map $T_0\psi$ of this orthogonal complement is a $G_{m_e,\xi}$-invariant subspace of X; we choose this subspace as the vector space V in Definition 2.1. The space W can analogously be chosen to be invariant under the $G_{m_e,\xi}$ action on X.

Given these choices of subspaces, the action of $G_{m_e,\xi}$ on M induces a well-defined action on $\mathfrak{m}^* \times V$ via the slice map. Equivariance of the momentum map, the coordinate chart, and the projection $\mathbb{P}_\mathfrak{m}$ imply that the slice map Ψ is equivariant with respect to this action. □

Recall that the relative equilibrium equations were obtained using two consecutive applications of the Implicit Function Theorem (Steps 1 and 2) followed by the Lyapunov–Schmidt reduction procedure (Step 3). It is well known that if the Implicit Function Theorem is applied to an equation $F = c$ determined by an equivariant map F and a fixed point c of the group action, then the resulting implicitly defined function is also equivariant. In addition, if the Lyapunov–Schmidt reduction procedure is applied to such an equation using invariant subspaces, then the resulting functions and equations will be equivariant. (See, e.g., [13,15] for precise statements and proofs of these results). Using these fundamental results, we now show that, given appropriate choices of slice maps and subspaces, the generator map Ξ and the functions B and ρ determining the reduced relative equilibrium equations are equivariant with respect to the induced $G_{m_e,\xi}$ action on $\mathfrak{m}^* \times V \times \mathfrak{g}_{m_e}$.

Proposition 3.3. *If the spaces $\mathfrak{m}$, $\mathfrak{q}$, V, and W are $G_{m_e,\xi}$ invariant and the slice mapping is $G_{m_e,\xi}$-equivariant, then the maps Ξ, v_1, B, ρ, and F are all $G_{m_e,\xi}$-equivariant.*

Proof. It suffices to show that the functions F_1, F_2, and F_3 given in Steps 1–3 are $G_{m_e,\xi}$-equivariant. We first consider the mapping $F_1 : \mathcal{U} \times \mathfrak{g}_{m_e} \times \mathfrak{m} \times \mathfrak{q} \to \mathfrak{q}^*$ introduced in Step 1. For arbitrary $g \in G_{m_e,\xi}$:

$$\begin{aligned} F_1\big(g \cdot (\eta, v, \alpha, \beta, \gamma)\big) &= i_\mathfrak{q}^* \,\mathrm{ad}^*_{(\xi + g\cdot\alpha + g\cdot\beta + g\cdot\gamma)} \mathbf{j}(g\cdot\eta, g\cdot v) \\ &= i_\mathfrak{q}^* \,\mathrm{ad}^*_{\mathrm{Ad}_g(\xi+\alpha+\beta+\gamma)} \mathrm{Ad}^*_{g^{-1}} \mathbf{j}(\eta, v) \\ &= i_\mathfrak{q}^* \,\mathrm{Ad}^*_{g^{-1}}\big(\mathrm{ad}^*_{(\xi+\alpha+\beta+\gamma)} \mathbf{j}(\eta, v)\big) \end{aligned}$$

$$= \mathrm{Ad}^*_{g^{-1}}\big(i^*_{\mathfrak{q}}\,\mathrm{ad}^*_{(\xi+\alpha+\beta+\gamma)}\,\mathbf{j}(\eta, v)\big)$$
$$= g \cdot F_1(\eta, v, \alpha, \beta, \gamma).$$

Thus F_1 is $G_{m_e,\xi}$-equivariant and, hence, the implicitly defined functions γ and ω_1 are also $G_{m_e,\xi}$-equivariant. An analogous verification can be carried out for the mapping F_2 in Step 2, allowing us to conclude that the function ω_2 is also $G_{m_e,\xi}$-equivariant.

To establish the invariance (respectively equivariance) of the spaces and maps constructed in Step 3, we first note that $\mathcal{H}^\xi$ is $G_{m_e,\xi}$-invariant, since the augmented Hamiltonian h^ξ is G_ξ-invariant and the slice map Ψ is $G_{m_e,\xi}$-equivariant. Equivariance of the map F, and hence invariance of the subspaces $\ker F$ and $\operatorname{range} F$, follows immediately from the invariance of $\mathcal{H}^\xi$. The compactness of the group $G_{m_e,\xi}$ allows us to choose $G_{m_e,\xi}$-invariant complements V_1 and V_2 to $\ker F$ and $\operatorname{range} F$. (See, for instance, [15, Proposition 2.1].) With these choices, the canonical projection $\mathbb{P}$ and the function F_3 are equivariant. Consequently the function v_1, as well as the generator map Ξ and the reduced relative equilibrium equations are equivariant, as required. □

3.2. Treatment of the rigid residual equation

In this section we consider some situations in which the rigid residual map is either trivial or can be greatly simplified by using an appropriate slice mapping. For example, if G is Abelian, then the full rigid equation $\mathrm{ad}^*_\xi\,\mathbf{J}(m_e) = 0$ is trivial. Hence, the rigid residual equation is obviously satisfied. If G is not Abelian, but an appropriate invariance condition is satisfied, then there is a slice map $\Psi : \mathfrak{m}^* \times V \to M$ yielding a residual rigid equation involving only the Lie bracket on the isotropy subalgebra $\mathfrak{g}_\mu$. If $\mathfrak{g}_\mu$ is Abelian, this choice of slice map yields solutions of the residual rigid equation. We will present a few cases in which these helpful choices are possible.

Given a relative equilibrium m_e with momentum $\mu := \mathbf{J}(m_e)$, let $\mathcal{O}_\mu \subset \mathfrak{g}^*$ be the coadjoint orbit through μ, with tangent space

$$T_\mu\mathcal{O}_\mu = \big\{\mathrm{ad}^*_\zeta\mu \;\big|\; \zeta \in \mathfrak{g}\big\}$$

at μ. We shall say that a subspace $\mathfrak{q} \subset \mathfrak{g}$ is $\mathfrak{g}_\mu$-invariant if $[\mathfrak{g}_\mu, \mathfrak{q}] \subset \mathfrak{q}$.

We now prove that, generically, the rigid equation ρ can be reduced by an appropriate choice of slice map to an equation on $\mathfrak{g}_\mu$.

Proposition 3.4. *If the complement $\mathfrak{q}$ to $\mathfrak{g}_\mu$ in $\mathfrak{g}$ is $\mathfrak{g}_\mu$-invariant, then given any slice map $\Psi : \mathcal{U} \to M$ at m_e, there exists a map $\phi : \widetilde{\mathcal{U}} \to \mathfrak{q}$ such that*

(1) *the map $\widetilde{\Psi} : \widetilde{\mathcal{U}} \subset \mathcal{U} \to M$ given by*

$$\widetilde{\Psi}(\eta, v) = \exp\big(\phi(\eta, v)\big) \cdot \Psi(\eta, v) \tag{13}$$

is also a slice map,

(2) *the associated generator map $\widetilde{\Xi}$ takes values in $\mathfrak{g}_\mu$,*

(3) *the pullback* $\tilde{\mathbf{j}} := \mathbf{J} \circ \widetilde{\Psi}$ *of the momentum map takes values in* $\mu + \mathfrak{q}^\circ$,
(4) $\phi(0,0) = 0$ *and* $D\phi(0,0) = 0$.

If the original slice mapping is $G_{m_e,\xi}$*-equivariant, then* $\widetilde{\Psi}$ *is equivariant.*

Proof. We obtain the map ϕ through yet another application of the Implicit Function Theorem. Define $C : \mathfrak{m}^* \times V \times \mathfrak{q} \to \mathfrak{q}^*$ by

$$C(\eta, v, \phi) = i_{\mathfrak{q}}^* \big(\mathbf{J}\big(\exp(\phi) \cdot \Psi(\eta, v)\big) - \mu\big), \tag{14}$$

with differential

$$\begin{aligned} DC(\mathbf{0})(\delta\eta, \delta v, \delta\phi) &= i_{\mathfrak{q}}^* D\mathbf{J}(m_e)\big(T_{(0,0)}\Psi(\delta\eta, \delta v) + \delta\phi_M(m_e)\big) \\ &= i_{\mathfrak{q}}^*\big(\mathbb{P}_{\mathfrak{m}}^* \delta\eta - \operatorname{ad}_{\delta\phi}^* \mu\big) = -i_{\mathfrak{q}}^* \operatorname{ad}_{\delta\phi}^* \mu \end{aligned}$$

for arbitrary $\delta\eta \in \mathfrak{m}^*$, $\delta v \in V$, and $\delta\phi \in \mathfrak{q}$. Here (SM3), equivariance of the momentum map, and the identity $i_{\mathfrak{q}}^* \mathbb{P}_{\mathfrak{m}}^* = (\mathbb{P}_{\mathfrak{m}} \circ i_{\mathfrak{q}})^* = 0$ have been used to simplify the expressions. Since $\eta \mapsto i_{\mathfrak{q}}^* \operatorname{ad}_\eta^* \mu$ is an isomorphism from $\mathfrak{q}$ to $\mathfrak{q}^*$, the Implicit Function Theorem implies that there is a neighborhood $\widetilde{\mathcal{U}}$ of $(0,0)$ in $\mathfrak{m}^* \times V$ and a function $\phi : \widetilde{\mathcal{U}} \to \mathfrak{q}$ such that $\phi(0,0) = 0$, $D\phi(0,0) = 0$, and $C(\eta, v, \phi(\eta, v)) = 0$.

Using $\phi : \widetilde{\mathcal{U}} \subset \mathfrak{m}^* \times V \to \mathfrak{q}$ and (13), we see that the pullback $\tilde{\mathbf{j}}$ of the momentum map satisfies

$$i_{\mathfrak{q}}^*\big(\operatorname{ad}_{\xi+\alpha+\beta}^* \tilde{\mathbf{j}}(\eta, v)\big) = i_{\mathfrak{q}}^*\big(\operatorname{ad}_{\xi+\alpha+\beta}^*\big(\tilde{\mathbf{j}}(\eta, v) - \mu\big)\big) = 0$$

for all $(\eta, v, \alpha, \beta) \in \widetilde{\mathcal{U}}_1$. Thus executing Step 1 of Section 3 using the modified slice mapping $\widetilde{\Psi}$ yields a mapping $\tilde{\gamma} : \widetilde{\mathcal{U}}_1 \subset \mathfrak{m}^* \times V \times \mathfrak{g}_{m_e} \times \mathfrak{m} \to \mathfrak{q}$ satisfying

$$\begin{aligned} 0 &= F_1\big(\eta, v, \alpha, \beta, \tilde{\gamma}(\eta, v, \alpha, \beta)\big) = i_{\mathfrak{q}}^*\big(\operatorname{ad}_{\xi+\alpha+\beta+\tilde{\gamma}(\eta,v,\alpha,\beta)}^* \tilde{\mathbf{j}}(\eta, v)\big) \\ &= i_{\mathfrak{q}}^*\big(\operatorname{ad}_{\tilde{\gamma}(\eta,v,\alpha,\beta)}^* \tilde{\mathbf{j}}(\eta, v)\big) \end{aligned}$$

for any $(\eta, v, \alpha, \beta) \in \widetilde{\mathcal{U}}_1$. $\tilde{\gamma} \equiv 0$ clearly satisfies this equation; hence it is the unique solution of the equation $F_1 \equiv 0$ given by the Implicit Function Theorem. Thus Steps 2 and 3 yield the generator map

$$\widetilde{\Xi}(\eta, v_0, \alpha) = \xi + \alpha + \beta(\eta, v_0 + v_1(\eta, v_0, \alpha), \alpha) \in \mathfrak{g}_\mu.$$

Suppose now that the slice map Ψ satisfies the property (ESM). Note that for any $(\eta, v, \phi) \in \mathfrak{m}^* \times V \times \mathfrak{q}$ and any $h \in G_{m_e,\xi} \subset G_\mu$

$$\begin{aligned} C(h \cdot \eta, h \cdot v, h \cdot \phi) &= i_{\mathfrak{q}}^*\big(\mathbf{J}\big(\exp(h \cdot \phi) \cdot \Psi(h \cdot \eta, h \cdot v)\big) - \mu\big) \\ &= h \cdot i_{\mathfrak{q}}^*\big(\mathbf{J}\big(\exp(\phi) \cdot \Psi(\eta, v)\big) - \mu\big) = h \cdot C(\eta, v, \phi). \end{aligned}$$

Equivariance of C implies that ϕ, and hence $\widetilde{\Psi}$, are equivariant. □

If the hypotheses of Proposition 3.4 are satisfied, the rigid residual equation involves only elements of $\mathfrak{g}_\mu$ and $\mathfrak{g}_\mu^*$. Specifically, if we let $[\ ,\ ]_\mu$ denote the Lie bracket on $\mathfrak{g}_\mu$ and $\mathbf{J}_\mu : M \to \mathfrak{g}_\mu^*$ denote the momentum map associated to the action of G_μ on M, namely $\mathbf{J}_\mu = i_{\mathfrak{g}_\mu}^* \mathbf{J}$, then ρ satisfies

$$\langle \rho(\eta, v_0, \alpha), \beta \rangle = \langle \mathbf{J}_\mu \big(\widetilde{\Psi} \big(\eta, v_0 + v_1(\eta, v_0, \alpha) \big) \big), \big[\widetilde{\Xi}(\eta, v_0, \alpha), \beta \big]_\mu \rangle, \tag{15}$$

for all $\beta \in \mathfrak{m}$. In particular, if $\mathfrak{g}_\mu$ is Abelian, then ρ is identically zero. Thus we have established the following corollary.

Corollary 3.5. *Let m_e be a relative equilibrium with momentum $\mu = \mathbf{J}(m_e)$. If $\mathfrak{g}_\mu$ is Abelian and there exists a $\mathfrak{g}_\mu$-invariant complement to $\mathfrak{g}_\mu$ in $\mathfrak{g}$, then there is a slice map with respect to which the rigid residual map ρ is identically zero.*

Another approach to the search for solutions of the rigid residual equation is to restrict this search to fixed point subspaces corresponding to subgroups of the symmetry group of ρ. More explicitly, suppose that the hypotheses of Proposition 3.4 are satisfied and that we start with an equivariant slice map Ψ. In that case, Proposition 3.3 guarantees that ρ is $G_{m_e,\xi}$-equivariant and satisfies (15). Equivariance implies that for any Lie subgroup $K \subset G_{m_e,\xi}$, the map ρ maps the set of fixed points of K into the set of fixed points of K in $\mathfrak{m}^*$. Hence all the zeroes of the restriction

$$\rho^K : (\mathfrak{m}^*)^K \times V_0^K \times \mathfrak{g}_{m_e}^K \to (\mathfrak{m}^*)^K,$$

of ρ to $(\mathfrak{m}^*)^K \times V_0^K \times \mathfrak{g}_{m_e}^K$ are also zeroes of ρ, where the superscript K denotes the subspace of K-fixed points with respect to the relevant action. In other words, we can look for the solutions of the rigid residual equation by searching the zeroes of its restrictions to different sets of K-fixed points, with K and arbitrary subgroup of $G_{m_e,\xi}$ which, in principle, should be easier, since the dimension of the system has been lowered without introducing additional complexity into the equations.

If the restriction of the Lie bracket of the Lie algebra $\mathfrak{g}_\mu$ to $\mathfrak{g}_\mu^K$ is trivial, then the entire subspace $(\mathfrak{m}^*)^K \times V_0^K \times \mathfrak{g}_{m_e}^K$ consists of solutions of the rigid residual equation. Indeed, for any $(\eta, v_0, \alpha) \in (\mathfrak{m}^*)^K \times V_0^K \times \mathfrak{g}_{m_e}^K$, if we let

$$\nu = \mathbf{J}_\mu \big(\widetilde{\Psi} \big(\eta, v_0 + v_1(\eta, v_0, \alpha) \big) \big) \quad \text{and} \quad \zeta = \widetilde{\Xi}(\eta, v_0, \alpha),$$

then

$$\rho(\eta, v_0, \alpha) = \langle \nu, [\zeta, \cdot]_{\mathfrak{g}_\mu} \rangle.$$

The equivariance of $\widetilde{\Xi}$ and $\mathbf{J}_\mu$ implies that $\zeta \in \mathfrak{g}_\mu^K$ and $\nu \in (\mathfrak{g}_\mu^*)^K$. Also, since $\mathfrak{m} \subset \mathfrak{g}_\mu$, we have $(\mathfrak{m})^K \subset (\mathfrak{g}_\mu)^K$. Therefore, since $(\mathfrak{m}^*)^K \simeq (\mathfrak{m}^K)^*$, for any $\xi \in \mathfrak{m}^K$ we have

$$\langle \rho(\eta, v_0, \alpha), \xi \rangle = \langle \nu, [\zeta, \xi]_{\mathfrak{g}_\mu^K} \rangle = 0,$$

due to the hypothesis on the Lie bracket on $\mathfrak{g}_\mu^K$. The arbitrary character of $\xi \in \mathfrak{m}^K$ implies that $\rho(\eta, v_0, \alpha) = 0$.

Thus we have then proved the following

Proposition 3.6. *Let m_e be a relative equilibrium with momentum $\mu = \mathbf{J}(m_e)$ and generator $\xi \in \mathfrak{g}$. If there exists a $\mathfrak{g}_\mu$-invariant complement to $\mathfrak{g}_\mu$ in $\mathfrak{g}$, then for any subgroup $K \subset G_{m_e,\xi}$ for which the restriction of the Lie bracket of the Lie algebra $\mathfrak{g}_\mu$ to the set of fixed points $\mathfrak{g}_\mu^K$ is trivial, there is a slice map $\widetilde{\Psi}$ with respect to which the entire subspace $(\mathfrak{m}^*)^K \times V_0^K \times \mathfrak{g}_{m_e}^K$ consists of zeroes of the rigid residual equation ρ.*

(See [39] for persistence results on nondegenerate Hamiltonian relative equilibria valid under conditions of this sort.)

4. Persistence in Hamiltonian systems with Abelian symmetries

In this section we focus on the relative equilibria of Hamiltonian systems for which the symmetry group G is Abelian and the G action is proper. Let $m_e \in M$ be a relative equilibrium with generator ξ and momentum $\mu = \mathbf{J}(m_e)$. Since the adjoint and coadjoint actions of an Abelian group are trivial, $G_\mu = G$ and the rigid residual equation (B1) is trivially satisfied. We also assume that the bifurcation equation (B2) is trivial, i.e., that m_e is a *nondegenerate relative equilibrium*, with

$$\ker D^2 h^\xi(m_e) = \mathfrak{g}_\mu \cdot m_e = \mathfrak{g} \cdot m_e.$$

In this situation Steps 1 through 3 in Section 3 guarantee the existence of a $\mathfrak{m}^* \times \mathfrak{g}_{m_e}$-parameter family of relative equilibria *persisting* from m_e, whose dimension and structure we now study. We use the word *persistence* as opposed to the word *bifurcation*, given that the latter is customarily used to indicate a qualitative change in the family of relative equilibria as a given parameter is varied. This is analytically reflected in the need for a nontrivial Lyapunov–Schmidt reduction procedure in order to write the bifurcation equations. We shall see that in the case at hand no such tool will be necessary.

In this section we will use a very special slice mapping based on the Marle–Guillemin–Sternberg normal form [16,17,28] (we will refer to it as the *MGS-normal form*), that we briefly describe. The following exposition includes without proof the details of the MGS-normal form that will be needed in our discussion. For additional information the reader should consult the above mentioned original papers or [33,35,39].

We start by introducing the main ingredients of the MGS construction. Even though we are concerned here only with the Abelian case, we present the general definition. First, the properness of the G-action implies that the isotropy subgroup G_{m_e} is compact. Second, the vector space $V_{m_e} := (\mathfrak{g} \cdot m_e)^\omega / ((\mathfrak{g} \cdot m_e)^\omega \cap (\mathfrak{g} \cdot m_e)) = (\ker D\mathbf{J}(m_e))/(\mathfrak{g}_\mu \cdot m_e)$ is called the *symplectic normal space*. (Here $(\mathfrak{g} \cdot m_e)^\omega$ denotes the symplectic orthogonal complement to $\mathfrak{g} \cdot m_e$.) V_{m_e} is a symplectic vector space with the symplectic normal form $\omega_{V_{m_e}}$ defined by

$$\omega_{V_{m_e}}\big([v], [w]\big) := \omega(m_e)(v, w)$$

for any v and $w \in \ker D\mathbf{J}(m_e)$. The mapping $(h,[v]) \mapsto [h \cdot v]$, with $h \in G_{m_e}$ and $[v] \in V_{m_e}$, defines a canonical action of the Lie group G_{m_e} on $(V_{m_e}, \omega_{V_{m_e}})$, where $g \cdot u$ denotes the tangent lift of the G-action on TM, for $g \in G$ and $u \in TM$.

For simplicity of notation, we shall set $H = G_{m_e}$, $N = V_{m_e}$, and drop the brackets [ ] indicating the equivalence classes in N, simply writing $v \in N$ for the remainder of the section. The canonical H-action on N is linear by construction and globally Hamiltonian with momentum map $\mathbf{J}_N : N \to \mathfrak{h}^*$ given by

$$\big\langle \mathbf{J}_N(v), \eta \big\rangle = \tfrac{1}{2}\omega_N\big(\eta_N(v), v\big),$$

for arbitrary $\eta \in \mathfrak{h}^*_{m_e}$ and $v \in N$. Here η_N denotes the infinitesimal generator on N associated to the algebra element $\eta \in \mathfrak{h}$.

The MGS-normal form is based on the construction of a model space Y for M, with symplectic structure ω_Y, that we introduce in the following proposition.

Proposition 4.1. *Let $m_e \in M$ and $\mu = \mathbf{J}(m_e)$. Let (N, ω_N) be the symplectic normal space at m_e. Consider the inclusions $\mathfrak{m}^* \subset \mathfrak{g}^*_\mu \subset \mathfrak{g}^*$ relative to an Ad_H-invariant inner product on $\mathfrak{g}$. Then the manifold*

$$Y := G \times_H (\mathfrak{m}^* \times N)$$

can be endowed with a symplectic structure ω_Y with respect to which the left G-action $g \cdot [h, \eta, v] = [gh, \eta, v]$ on Y is globally Hamiltonian with momentum map $\mathbf{J}_Y : Y \to \mathfrak{g}^*$ given by

$$\mathbf{J}_Y\big([g, \rho, v]\big) = \mathrm{Ad}^*_{g^{-1}}\big(\mu + \rho + \mathbf{J}_N(v)\big). \tag{16}$$

Theorem 4.2 (Marle–Guillemin–Sternberg normal form). *For any $m_e \in M$, the manifold*

$$Y := G \times_H (\mathfrak{m}^* \times N)$$

introduced in Proposition 4.1 *is a Hamiltonian G-space and there are G-invariant neighborhoods U of m_e in M, U' of $[e, 0, 0]$ in Y, and an equivariant symplectomorphism $\phi : U \to U'$ satisfying $\phi(m_e) = [e, 0, 0]$ and $\mathbf{J}_Y \circ \phi = \mathbf{J}$.*

Since we intend to prove general statements about relative equilibria of Hamiltonian systems with Abelian symmetries, the previous theorem allows us to reduce the problem to the study of systems of the form (Y, ω_Y). Indeed, we will assume that the MGS-normal form is constructed around the relative equilibrium m_e represented by $[e, 0, 0]$ in "MGS coordinates." It can be easily shown that the map given by

$$\begin{aligned} \Psi : \mathfrak{m}^* \times N &\to Y \\ (\eta, v) &\mapsto [e, \eta, v] \end{aligned} \tag{17}$$

is a slice mapping at the point $[e, 0, 0]$ for Dh_Y^{ξ}, where h_Y^{ξ} is the representation of h^{ξ} given by the MGS-normal form.

Before stating the following theorem, we recall from elementary differential geometry the basic notion of the *rank of a surface* given in parametric form. Let $g : U \subset \mathbb{R}^n \to \mathbb{R}^m$ be a parameterization of a surface S in $\mathbb{R}^m$. Given a value $u \in \mathbb{R}^n$ of the parameter, the rank of the surface $S_{g(u)}$ at the point $g(u) \in \mathbb{R}^m$ is the rank of the Jacobian of the function g at u. If this rank is constant, the Fibration Theorem [2, Theorem 3.5.18] guarantees that S is a submanifold of $\mathbb{R}^m$ and its rank coincides with the dimension of S as a manifold on its own.

Theorem 4.3. *Let $m_e \in M$ be a nondegenerate relative equilibrium with generator $\xi \in \mathfrak{g}$. Set $H := G_{m_e}$ and $\mu = \mathbf{J}(m_e)$. Then there is a surface S of relative equilibria through m_e that can be locally expressed as*

$$S = \big\{\big[g, \eta, v(\eta, \alpha)\big] \in Y \;\big|\; g \in G,\ \eta \in \mathfrak{m}^*,\ \alpha \in \mathfrak{h}\big\},$$

using the MGS normal form Y constructed around the orbit $G \cdot m_e$. Here $v : \mathfrak{m}^* \times \mathfrak{h} \to N$ is a smooth function such that $v(0, 0) = 0$ and $\operatorname{rank}(Dv(\eta, \alpha)) = \dim H - \dim H_{v(\eta,\alpha)}$. The rank, $\operatorname{rank} S_{[g,\eta,v(\eta,\alpha)]}$, of the surface S at the relative equilibrium $[g, \eta, v(\eta, \alpha)]$ equals

$$\operatorname{rank} S_{[g,\eta,v(\eta,\alpha)]} = 2(\dim G - \dim H) + (\dim H - \dim H_{v(\eta,\alpha)}). \tag{18}$$

Proof. The surface S of relative equilibria is constructed in Steps 1 through 3 of Section 3, taking as slice mapping the map $\Psi(\eta, v) = [e, \eta, v]$ constructed with the help of the MGS-normal form. Indeed, since the nondegeneracy of m_e and the Abelian character of G imply that (B1) and (B2) are trivially satisfied, there is a neighborhood $\mathcal{U} \subset \mathfrak{m}^* \times \mathfrak{h}$ of the point $(0, 0)$ and functions $v : \mathcal{U} \to N$ and $\Xi : \mathcal{U} \to \mathfrak{g}$ such that for any $(\eta, \alpha) \in \mathcal{U}$, the point $[e, \eta, v(\eta, \alpha)] \in Y \simeq M$ is a relative equilibrium of the system (M, ω, h) with generator $\Xi(\eta, \alpha) \in \mathfrak{g}$. At the same time, since the Lie group G is Abelian and the Hamiltonian flow F_t associated to h is G-equivariant, it is easy to verify that if the point $[e, \eta, v(\eta, \alpha)]$ is a relative equilibrium with generator $\Xi(\eta, \alpha) \in \mathfrak{g}$ then, for any $g \in G$, the point $[g, \eta, v(\eta, \alpha)]$ is also a relative equilibrium with the same generator. In order to prove (18), we compute $Dv(\eta, \alpha)$ by implicit differentiation of the equation $F_3(\eta, v(\eta, \alpha), \alpha) = 0$ defining the function v in Step 3. Note that in this case the space V_0 is trivial and we have dropped the subscript from v_1. Note that $\mathfrak{q}$ is trivial in the Abelian case and hence

$$\omega_2(\eta, v, \alpha) = \omega_1\big(\eta, v, \alpha, \beta(\eta, v, \alpha)\big) = \xi + \alpha + \beta(\eta, v, \alpha).$$

For $u \in N$, for arbitrary $\delta\alpha \in \mathfrak{h}$, if we set $\alpha_t = \alpha + t\delta\alpha$, we have

$$\begin{aligned} 0 &= \big\langle D_N F_3\big(\eta, v(\eta, \alpha), \alpha\big)\big(Dv(\eta, \alpha)(0, \delta\alpha)\big), u\big\rangle \\ &= \frac{\mathrm{d}}{\mathrm{d}t} D\mathcal{H}^{\xi+\alpha_t \beta(\eta, v(\eta,\alpha_t),\alpha_t)}\big(\eta, v(\eta, \alpha_t)\big)(0, u)\Big|_{t=0} \\ &= D^2\mathcal{H}^{\xi+\alpha+\beta(\eta,v(\eta,\alpha),\alpha)}\big(\eta, v(\eta, \alpha)\big)\big(\big(0, Dv(\eta, \alpha)(0, \delta\alpha)\big), (0, u)\big) \\ &\quad - \big\langle D\mathbf{J}_N(\eta, \alpha)u, \delta\alpha\big\rangle. \end{aligned} \tag{19}$$

The last equality follows from the identity

$$\mathbf{j}(\eta, v) = \mu + \eta + \mathbf{j}_N(v),$$

which implies that

$$\big\langle D\mathbf{j}(\eta, v)(0, u), \delta\alpha + \delta\beta \big\rangle = \big\langle D\mathbf{J}_N(v)u, \delta\alpha + \delta\beta \big\rangle = \big\langle D\mathbf{J}_N(v)u, \delta\alpha \big\rangle.$$

By hypothesis, the quadratic form $D_{NN}\mathcal{H}^{\xi}(0,0)$ is nondegenerate; therefore, since nondegeneracy is an open condition, $D_{NN}\mathcal{H}^{\xi+\alpha+\beta(\eta,\alpha)}(\eta, v(\eta,\alpha))$ is nondegenerate for any $(\eta,\alpha) \in \mathfrak{m}^* \times \mathfrak{h}$ sufficiently close to $(0,0)$. Hence the rank of $D_{\mathfrak{h}} v(\eta,\alpha)$ equals the rank of $D\mathbf{J}_N(v(\eta,\alpha))$ at a point $(\eta,\alpha) \in \mathfrak{m}^* \times \mathfrak{h}$ sufficiently close to $(0,0)$. Thus

$$\begin{aligned} \operatorname{rank}\big(D_{\mathfrak{h}} v(\eta,\alpha)\big) &= \operatorname{rank}\big(D\mathbf{J}_N\big(v(\eta,\alpha)\big)\big) = \dim\big[(\mathfrak{h}_{v(\eta,\alpha)})^{\operatorname{ann}(\mathfrak{h}^*)}\big] \\ &= \dim H - \dim H_{v(\eta,\alpha)}, \end{aligned} \tag{20}$$

as required. In the previous expression the symbol $(\mathfrak{h}_{v(\eta,\alpha)})^{\operatorname{ann}(\mathfrak{h}^*)}$ denotes the annihilator of $\mathfrak{h}_{v(\eta,\alpha)}$ in $\mathfrak{h}^*$, as opposed to $\mathfrak{g}^*$.

The expression (18) for the rank of the surface S at a relative equilibrium $[g, \eta, v(\eta,\alpha)]$ is a straightforward consequence of the formula (20) for the rank of $D_{\mathfrak{h}} v(\eta,\alpha)$. The rank of S at $[g, \eta, v(\eta,\alpha)]$ is the rank of the parameterization

$$\begin{aligned} \mathcal{S}: G \times \mathfrak{m}^* \times \mathfrak{h} &\to G \times \mathfrak{m}^* \times N \to G \times_H (\mathfrak{m}^* \times N) \\ (g, \eta, \alpha) &\mapsto \big(g, \eta, v(\eta,\alpha)\big) \mapsto \big[g, \eta, v(\eta,\alpha)\big] \end{aligned}$$

of the surface S. The map $\mathcal{S}$ has rank

$$\begin{aligned} \operatorname{rank}(T_{(g,\alpha,\eta)}\mathcal{S}) &= \operatorname{rank}(S_{[g,\eta,v(\eta,\alpha)]}) = \dim G + \dim \mathfrak{m}^* + \operatorname{rank}\big(Dv(\alpha)\big) - \dim H \\ &= 2(\dim G - \dim H) + \dim H - \dim H_{v(\eta,\alpha)}, \end{aligned}$$

at $[g, \eta, v(\eta,\alpha)]$, as required. □

As a corollary to the previous theorem, we can formulate a generalization of a result due to Lerman and Singer [21], originally stated for toral actions, to proper actions of Abelian Lie groups. This result has already been presented in [33].

Corollary 4.4. *Under the hypotheses of Theorem 4.3, there is a symplectic manifold Σ of relative equilibria of h satisfying $m_e \in \Sigma$ and*

$$\dim \Sigma = 2(\dim G - \dim H).$$

Proof. The manifold Σ is the submanifold of the surface S, obtained by setting the parameter $\alpha \in \mathfrak{h}$ equal to zero; in other words

$$\Sigma = \bigl\{\bigl[g, \eta, v(\eta, 0)\bigr] \in Y \bigm| g \in G,\ \eta \in \mathfrak{m}^*\bigr\}. \tag{21}$$

The submanifold Σ is a smooth manifold, since (18) implies that it has constant rank $2(\dim G - \dim H)$; that is, the map

$$\begin{aligned} \mathcal{T} : G \times \mathfrak{m}^* \ &\to\ G \times \mathfrak{m}^* \times N \to G \times_H (\mathfrak{m}^* \times N) \\ (g, \eta) \ &\mapsto\ \bigl(g, \eta, v(\eta, 0)\bigr) \mapsto \bigl[g, \eta, v(\eta, 0)\bigr] \end{aligned}$$

with image Σ is a local constant rank map around $(e, 0) \in G \times \mathfrak{m}^*$ with rank equal to $2(\dim G - \dim H)$, which implies that the surface Σ is locally a manifold through the relative equilibrium m_e, of dimension $2(\dim G - \dim H)$. (See, for instance, [2, Theorem 3.5.18].)

The symplectic nature of Σ can be verified in a straightforward manner. Indeed, we will check that if $i : \Sigma \hookrightarrow Y$ is the natural inclusion then the pair (Σ, ω_Σ), with $\omega_\Sigma = i^*\omega_Y$, is a symplectic submanifold of (Y, ω_Y). Let $\pi : G \times \mathfrak{m}^* \times N \to G \times_H (\mathfrak{m}^* \times N)$ be the canonical projection. Note that every vector in $T_{[g,\eta,v(\eta,0)]}\Sigma$ can be written as $T_{(g,\eta,v(\eta,0))}\pi(T_eL_g \cdot \zeta, \delta\eta, D_{\mathfrak{m}^*}v(\eta, 0) \cdot \delta)$, for some $\zeta \in \mathfrak{g}$ and $\delta\eta \in \mathfrak{m}^*$. The two-form ω_Σ is clearly closed. In order to prove that it is nondegenerate, let us suppose that the vector $T_{(g,\eta,v(\eta,0))}\pi(T_eL_g \cdot \zeta, \delta\eta, D_{\mathfrak{m}^*}v(\eta, 0) \cdot \delta\eta')$ is such that

$$\begin{aligned} 0 \ = \ \omega_\Sigma\bigl(\bigl[g, \eta, v(\eta, 0)\bigr]\bigr)\bigl(&T_{(g,\eta,v(\eta,0))}\pi\bigl(T_eL_g\zeta, \delta\eta, D_{\mathfrak{m}^*}v(\eta, 0)\delta\eta\bigr), \\ &T_{(g,\eta,v(\eta,0))}\pi\bigl(T_eL_g\zeta', \delta\eta', D_{\mathfrak{m}^*}v(\eta, 0)\delta\eta'\bigr)\bigr) \end{aligned} \tag{22}$$

for every $\zeta' \in \mathfrak{g}$ and $\delta\eta' \in \mathfrak{m}^*$. We will show that this implies that

$$T_{(g,\eta,v(\eta,0))}\pi\bigl(T_eL_g\zeta, \delta\eta, D_{\mathfrak{m}^*}v(\eta, 0)\delta\eta\bigr) = 0.$$

Using $\omega_\Sigma = i^*\omega_Y$ and the explicit expression of the symplectic form ω_Y associated to the MGS normal form (see the previously quoted original papers, as well as [33,34,39]), we can write (22) in the form

$$\begin{aligned} 0 \ = \ &\bigl\langle \delta\eta' + D\mathbf{J}_N\bigl(v(\eta, 0)\bigr)\bigl(D_{\mathfrak{m}^*}v(\eta, 0)\delta\eta'\bigr), \zeta\bigr\rangle \\ &- \bigl\langle \delta\eta + D\mathbf{J}_N\bigl(v(\eta, 0)\bigr)\bigl(D_{\mathfrak{m}^*}v(\eta, 0)\delta\eta\bigr), \zeta'\bigr\rangle \\ &+ \omega_N\bigl(D_{\mathfrak{m}^*}v(\eta, 0)\delta\eta, D_{\mathfrak{m}^*}v(\eta, 0)\delta\eta'\bigr) \end{aligned}$$

for any $\delta\eta' \in \mathfrak{m}^*$ and $\zeta' \in \mathfrak{g}$. If we fix $\delta\eta' = 0$ and let ζ' be arbitrary, we obtain

$$\delta\eta + D\mathbf{J}_N\bigl(v(\eta, 0)\bigr) \cdot \bigl(D_{\mathfrak{m}^*}v(\eta, 0) \cdot \delta\eta\bigr) = 0.$$

Since $\delta\eta \in \mathfrak{m}^*$, $D\mathbf{J}_N(v(\eta,0)) \cdot (D_{\mathfrak{m}^*}v(\eta,0)\cdot\delta\eta) \in \mathfrak{h}^*$, and $\mathfrak{m}^* \cap \mathfrak{h}^* = \{0\}$, we have

$$\delta\eta = D\mathbf{J}_N\big(v(\eta,0)\big) \cdot \big(D_{\mathfrak{m}^*}v(\eta,0)\cdot\delta\eta\big) = 0. \tag{23}$$

If we now fix $\zeta' = 0$ and let $\delta\eta'$ be arbitrary, we obtain $\zeta \in \mathfrak{h}$, which, together with (23), guarantees that $T_{(g,\eta,v(\eta,0))}\pi(T_eL_g\zeta, \delta\eta, D_{\mathfrak{m}^*}v(\eta,0)\delta\eta) = 0$, as required. □

In the remainder of this section we will show that the persistence phenomena described by Theorem 4.3 and Corollary 4.4 preserve stability. More specifically, we will show that if the relative equilibrium m_e is stable, then the entire local symplectic manifold Σ given by Corollary 4.4 consists of stable relative equilibria. First, we recall the definition of nonlinear stability of a relative equilibrium:

Definition 4.5. Let G' be a subgroup of G. A relative equilibrium $m_e \in M$ is called *G'-stable*, or *stable modulo G'*, if for any G'-invariant open neighborhood V of the orbit $G' \cdot m_e$, there is an open neighborhood $U \subseteq V$ of m_e, such that if F_t is the flow of the Hamiltonian vector field X_h and $u \in U$, then $F_t(u) \in V$ for all $t \geqslant 0$.

Before recalling the stability result to be used here, we introduce the following notation. Suppose that we fix a splitting of $\mathfrak{g}$ as in (2). If $\xi = \xi_1 + \xi_2$, with $\xi_1 \in \mathfrak{g}_{m_e}$ and $\xi_2 \in \mathfrak{m}$, is a generator of the relative equilibrium m_e, then the unique element $\xi_2 \in \mathfrak{m}$ is called the *orthogonal generator* of m_e with respect to the splitting (2).

We now state the following theorem whose proof can be found in [21] or in [34].

Theorem 4.6. *Let $(M, \{\cdot,\cdot\}, h)$ be a Poisson system with a symmetry given by the Lie group G acting properly on M in a globally Hamiltonian fashion, with associated equivariant momentum map $\mathbf{J}: M \to \mathfrak{g}^*$. Assume that the Hamiltonian $h \in C^\infty(M)$ is G-invariant. Let $m_e \in M$ be a relative equilibrium such that $\mu = \mathbf{J}(m_e)$, $\mathfrak{g}^*$ admits an $\mathrm{Ad}^*_{G_\mu}$-invariant inner product, $H := G_{m_e}$, and $\xi \in \mathrm{Lie}(N_{G_\mu}(H))$ is its orthogonal generator, relative to a given Ad_H-invariant splitting. Let h^ξ denote the scalar function $h^\xi(m) := h(m) - \langle \mathbf{J}(m), \xi\rangle$. If the quadratic form $D^2h^\xi(m_e)|_{W\times W}$ is definite for some (and hence for any) subspace W such that*

$$\ker D\mathbf{J}(m_e) = W \oplus \mathfrak{g}_\mu \cdot m_e,$$

then m_e is a G_μ-stable relative equilibrium. If $\dim W = 0$, then m_e is always a G_μ-stable relative equilibrium. The quadratic form $D^2h^\xi(m_e)|_{W\times W}$ will be called the stability form *of the relative equilibrium m_e.*

A relative equilibrium satisfying the hypotheses of Theorem 4.6 is said to be *formally stable*. Note that in the Abelian case all the adjoint invariance requirements in the statement of the previous theorem are trivially satisfied. We now state our stability persistence result.

Proposition 4.7. *Under the conditions of Corollary* 4.4, *suppose that the relative equilibrium* m_e *is formally* (*and consequently nonlinearly*) *stable that is, it has an orthogonal generator* $\xi \in \mathfrak{m}$ *with respect to the splitting* (2) *such that the quadratic form*

$$D^2 h^\xi(m_e)|_{W\times W}$$

is definite for some (*and hence for any*) *subspace* W *such that*

$$\ker D\mathbf{J}(m_e) = W \oplus \mathfrak{g}\cdot m_e,$$

then the symplectic manifold Σ *of relative equilibria passing through* m_e *can be chosen* (*by taking, if necessary, a sufficiently small neighborhood of* m_e *in the submanifold* Σ *of Corollary* 4.4) *to consist exclusively of nonlinearly stable relative equilibria.*

Proof. Recall that the symplectic manifold Σ consists of points of the form $[g,\eta, v(\eta,0)] \in Y$, with $\eta \in \mathfrak{m}^*$ sufficiently close to 0, which are relative equilibria with generator $\xi + \beta(\eta,0)$. For simplicity of notation, we will write $v = v(\eta,0)$ and $\beta = \beta(\eta,0)$ for the remainder of the proof. Since $\xi \in \mathfrak{m}$ is by hypothesis an orthogonal generator with respect to the splitting (2) and $\beta \in \mathfrak{m}$, the generator $\xi + \beta$ is also an orthogonal generator for the relative equilibrium $[g,\eta,v] \in Y$. Hence, in order to prove the Proposition it suffices to show that the quadratic form

$$D^2 h^{\xi+\beta}\big([g,\eta,v]\big)\big|_{W_{[g,\eta,v]}\times W_{[g,\eta,v]}},$$

is definite for some subspace $W_{[g,\eta,v]}$ such that $\ker D\mathbf{J}([g,\eta,v]) = W_{[g,\eta,v]} \oplus T_{[g,\eta,v]}(G\cdot[g,\eta,v])$. Using the expression of the momentum map in the MGS-coordinates described in Proposition 4.1, it is easy to verify that

$$\ker D\mathbf{J}\big([g,\eta,v]\big) = \big(\mathfrak{g}\cdot[g,\eta,v]\big) \oplus T_{[e,\eta,v]}\Phi_g\big(T_{(\eta,v)}\Psi\big(\{0\}\times \ker D\mathbf{J}_N(v)\big)\big),$$

where Φ_g denotes the G-action in MGS coordinates (see Proposition 4.1) and Ψ is the slice mapping introduced in (17). This identity singles out the space $T_{[e,\eta,v]}\Phi_g(T_{(\eta,v)}\Psi(\{0\}\times \ker D\mathbf{J}_N(v)))$ as a choice for $W_{[g,\eta,v]}$. We are now in position to study the definiteness of the stability form of the relative equilibrium $[g,\eta,v]$, using as $W_{[g,\eta,v]}$ the space we just mentioned. Indeed,

$$\begin{aligned}
&D^2 h^{\xi+\beta}\big([g,\eta,v]\big)\big|_{W_{[g,\eta,v]}\times W_{[g,\eta,v]}}\\
&\quad = D^2 h^{\xi+\beta}\big([g,\eta,v]\big)\big|_{(T_{[e,\eta,v]}\Phi_g(T_{(\eta,v)}\Psi(\{0\}\times\ker D\mathbf{J}_N(v))))\times(\text{same})}\\
&\quad = D^2\big(h^{\xi+\beta}\circ\Phi_g\big)\big([e,\eta,v]\big)\big|_{(T_{(\eta,v)}\Psi(\{0\}\times\ker D\mathbf{J}_N(v)))\times(\text{same})}\\
&\quad = D^2\mathcal{H}^{\xi+\beta}(\eta,v)\big|_{(\{0\}\times\ker D\mathbf{J}_N(v))\times(\text{same})}.
\end{aligned}$$

The formal stability of m_e implies that the quadratic form $D_{NN}\mathcal{H}^\xi(0,0)$ is definite, therefore, since definiteness is an open condition, for any $\eta \in \mathfrak{m}^*$ close enough to 0,

$D_{NN}\mathcal{H}^{\xi+\beta}(\eta, v)$ is also definite. Since $\ker D\mathbf{J}_N(v)$ is a subset of N, the definiteness of the stability form of relative equilibrium $[g, \eta, v]$ is guaranteed for small enough $\eta \in \mathfrak{m}^*$, as required. □

5. Bifurcation of relative equilibria with maximal isotropy

As in the previous section, we assume that G acts properly on M. However, we now assume that the relative equilibrium m_e is degenerate; that is, there is a generator $\xi \in \mathfrak{g}$ and a nontrivial vector subspace $V_0 \subset T_{m_e}M$ for which

$$\ker D^2 h^\xi(m_e) = \mathfrak{g}_\mu \cdot m_e \oplus V_0. \tag{24}$$

This hypothesis implies that the Lyapunov–Schmidt reduction used in the construction of the reduced critical point equations will be nontrivial and there will be the possibility of genuine bifurcation. In this section we will focus on the study of the bifurcation equation (B2); that is, we will assume that the rigid residual equation is satisfied and therefore the relative equilibria near m_e correspond to the zeroes of (B2).

In the framework of general dynamical systems, the bifurcation of relative equilibria with isotropy group K, out of a degenerate (i.e., nonhyperbolic) isolated equilibrium, is *generic*[2] if K is maximal and satisfies an additional property, e.g., has an odd-dimensional fixed-point subspace in the space V_0 on which the bifurcation equation is defined, or has an even dimensional fixed-point subspace together with a nontrivial S^1 action. The famous Equivariant Branching Lemma (see, e.g., [15]) belongs to the former case, while the latter appears in a work of Melbourne (see [8,30]). We shall see that both results have a counterpart in the symmetric Hamiltonian case, although being Hamiltonian is a nongeneric property from the general dynamical systems point of view. When searching for relative equilibria, the generator ($\alpha \in \mathfrak{g}$) or momentum ($\eta \in \mathfrak{g}^*$) serves as a bifurcation parameter, in addition to any physical control parameters present in system. Due to the "rigidity" of these geometric "parameters," care must be taken when adapting the bifurcation theorems to relative equilibria of Hamiltonian systems. As a final preliminary remark, we point out the fact that our theorems will be stated for bifurcation from a general relative equilibrium, not just from a pure (isolated) equilibrium. In the latter case, the gradient character of the bifurcation equation (see Remark 3.1) simplifies the arguments (see Remark 5.5).

5.1. A Hamiltonian equivariant branching lemma

In the situation described above, let $m_e \in M$ be a relative equilibrium satisfying the degeneracy hypothesis (24). As we saw in Proposition 3.3, the bifurcation equation (B2) can be constructed so as to be $G_{m_e,\xi}$-equivariant, which implies that for any subgroup

[2] Loosely speaking, a property of a system is generic if it is true unless additional constraints are added to the system (see [15]).

$K \subset G_{m_e,\xi}$, B can be restricted to the K-fixed point subspaces in its domain and range; hence we can find solutions of B by finding the solutions of

$$B^K := B|_{(\mathfrak{m}^*)^K \times V_0^K \times \mathfrak{g}_{m_e}^K} : (\mathfrak{m}^*)^K \times V_0^K \times \mathfrak{g}_{m_e}^K \to V_2^K.$$

Assume now that $K \subset G_{m_e,\xi}$ is a maximal isotropy subgroup of the $G_{m_e,\xi}$-action on V_0 and, moreover, that $\dim(V_0^K) = 1$. Under this hypothesis we will look for pairs $(\eta, v_0) \in (\mathfrak{m}^*)^K \times V_0^K$ satisfying

$$B^K(\eta, v_0, 0) = 0. \tag{25}$$

Note that $\dim(V_0^K) = 1$ implies that (see, for instance, [4])

$$L := N_{G_{m_e,\xi}}(K)/K \simeq \begin{cases} \{\mathrm{Id}\}, \\ \mathbb{Z}_2. \end{cases}$$

Recall that L acts naturally on $(\mathfrak{m}^*)^K$ and on V_0^K, and that B^K is L-equivariant. Depending on the character of the L-action, the first terms in the Taylor expansion of (25) can be written as

$$B^K(\eta, v_0, 0) = \begin{cases} \kappa \cdot \eta + v_0^2 c + \cdots = 0 & \text{if } L \simeq \{\mathrm{Id}\}, \\ v_0\big(\kappa \cdot \eta + v_0^2 c + \cdots\big) = 0 & \text{if } L \simeq \mathbb{Z}_2, \end{cases}$$

for some vector $\kappa \in (\mathfrak{m}^*)^K$ and some constant c that are generically nonzero. These expressions allow us to solve generically in both instances v_0 in terms of the other variables via the Implicit Function Theorem, giving us saddle-node type branches if $L \simeq \{\mathrm{Id}\}$ and a pitchfork bifurcation if $L \simeq \mathbb{Z}_2$ (see [15] for arguments of this sort). More explicitly, we have proved the following result.

Theorem 5.1 (Equivariant Branching Lemma). *Let $m_e \in M$ be a relative equilibrium of the Hamiltonian system $(M, \omega, h, G, \mathbf{J} : M \to \mathfrak{g}^*)$, where the Lie group G acts properly on the manifold M. Suppose that there is a generator $\xi \in \mathfrak{g}$ and a nontrivial vector subspace $V_0 \subset T_{m_e}M$ for which*

$$\ker D^2 h^\xi(m_e) = \mathfrak{g}_\mu \cdot m_e \oplus V_0.$$

Then, generically, for any subgroup $K \subset G_\xi \cap G_{m_e}$ for which $\dim(V_0^K) = 1$ and the rigid residual equation is satisfied on $(\mathfrak{m}^*)^K \times V_0^K \times \{0\}$, a branch of relative equilibria with isotropy subgroup K bifurcates from m_e. If $N_{G_{m_e,\xi}}(K)/K \simeq \{\mathrm{Id}\}$, the bifurcation is a saddle-node; if $N_{G_{m_e,\xi}}(K)/K \simeq \mathbb{Z}_2$, it is a pitchfork.

We will illustrate this result with an example in the following section.

5.2. Bifurcation with maximal isotropy of complex type

In what follows we will use a strategy similar to the one introduced by Melbourne [30] in the study of general equivariant dynamical systems to drop the hypothesis on the dimension of V_0^K in the Equivariant Branching Lemma. Our setup will be the same as in Theorem 5.1, but in this case we will be looking at maximal complex isotropy subgroups K of the $G_{m_e,\xi}$-action on V_0, that is, maximal isotropy subgroups K for which

$$L := N_{G_{m_e,\xi}}(K)/K \simeq \begin{cases} S^1, \\ S^1 \times \mathbb{Z}_2. \end{cases} \tag{26}$$

Note that in such cases V_0^K has even dimension.

As in the previous section, we will use the equivariance properties of the bifurcation equation in order to restrict the search for its solutions to the K-fixed space $(\mathfrak{m}^*)^K \times V_0^K \times \mathfrak{g}_{m_e}^K$. Moreover, we will consider only solutions of the form $(0, v_0, \alpha) \in (\mathfrak{m}^*)^K \times V_0^K \times \mathfrak{p}$, where $\mathfrak{p}$ is some $\mathrm{Ad}_{N_{G_{m_e,\xi}}(K)}$-invariant complement to $\mathfrak{k}$ in $\mathrm{Lie}(N_{G_{m_e,\xi}}(K))$. Note that (26) implies that $\mathfrak{p} \simeq \mathfrak{l} \simeq \mathbb{R}$.

We now show that the adjoint action of $N_{G_{m_e,\xi}}(K)$ on $\mathfrak{p}$ is trivial. The canonical projection $\pi : N_{G_{m_e,\xi}}(K) \to L$ is a group homomorphism; hence the commutativity of L implies that

$$\pi\left(ghg^{-1}\right) = \pi(g)\pi(h)\pi(g)^{-1} = \pi(h)$$

for any $g, h \in N_{G_{m_e,\xi}}(K)$. In particular,

$$T_e\pi \cdot (\mathrm{Ad}_g\alpha) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \pi\left(g \exp(t\alpha) g^{-1}\right) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \pi \exp(t\alpha) = T_e\pi \cdot \alpha$$

for any $g \in N_{G_{m_e,\xi}}(K)$ and $\alpha \in \mathrm{Lie}(N_{G_{m_e,\xi}}(K))$, which implies that $\mathrm{Ad}_g - \mathrm{id}$ maps $\mathrm{Lie}(N_{G_{m_e,\xi}}(K))$ into $\ker(T_e\pi) = \mathfrak{k}$. Since $\mathfrak{p} \cap \mathfrak{k} = \{0\}$ and $\mathfrak{p}$ is $\mathrm{Ad}_{N_{G_{m_e,\xi}}(K)}$-invariant, it follows that $(\mathrm{Ad}_g - \mathrm{id})|_{\mathfrak{p}} = 0$ for all $g \in N_{G_{m_e,\xi}}(K)$, i.e., that the adjoint action on $\mathfrak{p}$ is trivial.

Theorem 5.2. *Let $m_e \in M$ be a relative equilibrium of the Hamiltonian system $(M, \omega, h, G, \mathbf{J} : M \to \mathfrak{g}^*)$, where the Lie group G acts properly on the manifold M. Suppose that there is a generator $\xi \in \mathfrak{g}$ and a nontrivial vector subspace $V_0 \subset T_{m_e}M$ for which*

$$\ker D^2 h^\xi(m_e) = \mathfrak{g}_\mu \cdot m_e \oplus V_0.$$

Suppose that the fixed point set $V_0^{G_{m_e,\xi}} = \{0\}$. Then for each maximal complex isotropy subgroup K of the $G_{m_e,\xi}$-action on V_0 such that

$$\left[\mathrm{Lie}\left(N_{G_{m_e,\xi}}(K)\right), \mathfrak{g}_{m_e}^K\right] = 0$$

and each $\mathrm{Ad}_{N_{G_{m_e,\xi}}(K)}$*-invariant complement* $\mathfrak{p}$ *to* $\mathfrak{k}$ *in* $\mathrm{Lie}(N_{G_{m_e,\xi}}(K))$ *such that the rigid residual equation* $\rho(0, v_0, \alpha) = 0$ *is satisfied for all* $v_0 \in V_0^K$ *and* $\alpha \in \mathfrak{p}$*, there are generically at least* $\frac{1}{2}\dim V_0^K$ *(respectively* $\frac{1}{4}\dim V_0^K$*) branches of relative equilibria bifurcating from* m_e *if* $N_{G_{m_e,\xi}}(K)/K \simeq S^1$ *(respectively* $S^1 \times \mathbb{Z}_2$*).*

Proof. Let $B : \mathcal{U}_3 \subset \mathfrak{m}^* \times V_0 \times \mathfrak{g}_{m_e} \to V_2$ be the bifurcation equation corresponding to the reduced critical point equations constructed around m_e using the MGS-slice mapping introduced in (17). The equivariance of this slice mapping guarantees that B is $G_{m_e,\xi}$-equivariant; hence any solutions of

$$B^K := B|_{(\mathfrak{m}^*)^K \times V_0^K \times \mathfrak{g}_{m_e}^K} : (\mathfrak{m}^*)^K \times V_0^K \times \mathfrak{g}_{m_e}^K \to V_2^K$$

are solutions of B.

As we stated above, we will restrict our search to solutions in the set $\{0\} \times V_0^K \times \mathfrak{p}$, where $\mathfrak{p}$ is some $\mathrm{Ad}_{N_{G_{m_e,\xi}}(K)}$-invariant complement to $\mathfrak{k}$. Identify V_0 and V_2 using an invariant inner product $\langle\!\langle\ ,\ \rangle\!\rangle$ and define $\widetilde{B}^K : V_0^K \times \mathfrak{p} \to V_0^K$ through the relations

$$\big\langle\!\big\langle \widetilde{B}^K(v_0, \alpha), u \big\rangle\!\big\rangle := \big\langle B^K(0, v_0, \alpha), u \big\rangle = D_N \mathcal{H}^\eta\big(0, v_0 + v_1(0, v_0, \alpha)\big) u \big|_{\eta = \Xi(0, v_0, \alpha)} \tag{27}$$

for any v_0, $u \in V_0^K$ and $\alpha \in \mathfrak{p}$. The equivariance properties of B and the triviality of the action on $\mathfrak{p}$ imply that $\widetilde{B}^K$ satisfies the following equivariance condition:

$$\widetilde{B}^K(g \cdot v_0, \alpha) = g \cdot \widetilde{B}^K(v_0, \alpha) \quad \text{for all } g \in N_{G_{m_e,\xi}}(K). \tag{28}$$

Note that, as a corollary to this property, we have that

$$\widetilde{B}^K(0, \alpha) = 0 \quad \text{for all } \alpha, \tag{29}$$

since for all $g \in N_{G_{m_e,\xi}}(K)$, $g \cdot \tilde{B}^K(0, \alpha) = \widetilde{B}^K(0, \alpha)$ and, consequently, the isotropy subgroup of $\widetilde{B}^K(0, \alpha)$ contains $N_{G_{m_e,\xi}}(K)$ and hence it strictly contains K. The maximality of K as an isotropy subgroup implies that the isotropy subgroup of $\widetilde{B}^K(0, \alpha)$ is $G_{m_e,\xi}$. However, by hypothesis $V_0^{G_{m_e,\xi}} = \{0\}$; hence $\widetilde{B}^K(0, \alpha) = 0$, as claimed.

We find the solution branches by first finding an open ball $B_r(0)$ about the origin in V_0^K and a function $\alpha : B_r(0) \to \mathfrak{p}$ satisfying

$$\big\langle\!\big\langle \widetilde{B}^K\big(v_0, \alpha(v_0)\big), v_0 \big\rangle\!\big\rangle = 0,$$

then using $\widetilde{B}^K$ and α to define a family of vector fields on the unit sphere in V_0^K. Standard topological arguments show that these vector fields have the requisite number of equilibria, which correspond to solutions of the original equations.

As the first step in finding the function α, we compute the Taylor expansion of $\widetilde{B}^K$. As a result of the Lyapunov–Schmidt reduction and of (29), we can write

$$\widetilde{B}^K(v_0, \alpha) = L(\alpha) v_0 + g(v_0, \alpha),$$

where $L(\alpha)$ is a linear operator such that $L(0)=0$, and $g(v_0,\alpha)$ is such that $g(0,\alpha)=0$, $D_{v_0}g(0,\alpha)=0$ for all α. Moreover, a lengthy but straightforward computation shows that

$$L(\alpha) = -\mathbb{P}D_{NN}\mathbf{j}^{\alpha}(0,0) + L_1(\alpha),$$

where $\mathbf{j}^{\alpha} = \langle \mathbf{j}(\cdot,\cdot),\alpha\rangle$ and $L_1(0) = L_1'(0) = 0$. We now show that if we identify V_0 and V_2 by means of an invariant inner product and identify $\mathfrak{p}$ with $\mathbb{R}$, then there exists a constant $k \in \mathbb{N}^*$ such that

$$-\mathbb{P}D_{NN}\mathbf{j}^{\alpha}(0,0)|_{V_0^K} = \alpha k \mathbb{I}_{V_0^K}, \tag{30}$$

where $\mathbb{I}_{V_0^K}$ denotes the identity on V_0^K. Indeed, note that

$$\mathbf{j}^{\alpha}(0,v) = \big\langle \mathbf{J}\big(\Psi(0,v)\big),\alpha\big\rangle = \langle \mu,\alpha\rangle + \mathbf{J}_N^{\alpha}(v)$$

and hence

$$D_{NN}\mathbf{j}^{\alpha}(0,0)(v,w) = D_{NN}\mathbf{J}_N^{\alpha}(0)(v,w) = \omega_N\big(\alpha_N(v),w\big) \tag{31}$$

for any $v,w \in N$

We now restrict our attention to elements $v,\ w \in N^K$. Recall that since N is symplectic, the vector subspace N^K is symplectic with a canonical L action; hence for any $\alpha \in \mathfrak{l}$ and $v \in N$ there is an infinitesimally symplectic transformation A_α such that $\alpha_N(v) = A_\alpha v$. The equivariant version of the Williamson normal form due to Melbourne and Dellnitz [31], implies the existence of a basis in which A_α and ω_{N^K} admit simultaneous matrix representations consisting of three diagonal blocks corresponding to the subspaces $E_{\mathbb{R}}$, $E_{\mathbb{C}}$, and $E_{\mathbb{H}}$ of N^K on which L acts in a real, complex, and quaternionic fashion, respectively. Moreover, in this basis the restrictions of A_α and ω_{N^K} to $E_{\mathbb{C}}$ take the form:

$$\omega_{N^K}|_{E_{\mathbb{C}}} = \pm \mathrm{i}\mathbb{I} \quad \text{and} \quad A_\alpha|_{E_{\mathbb{C}}} = \pm \mathrm{i}\alpha\,\mathrm{diag}(k_1,\ldots,k_q)$$

for some natural numbers $k_1,\ldots,k_q$. The signs in these two equalities are consistent, that is, they are either both positive or both negative (in all that follows we will focus only on the positive case). These expressions follow directly from the tables in [31] and the absence of nilpotent parts in A_α, which is dictated by the requirement that A_α be the zero matrix when $\alpha = 0$. By hypothesis K is a maximal isotropy subgroup of the $G_{m_e,\xi}$-action on V_0 for which $V_0^K \subset E_{\mathbb{C}}$. Moreover, since the L-action on $V_0^K \setminus \{0\}$ is free, there exists $k \in \mathbb{N}^*$ such that

$$A_\alpha|_{V_0^K} = \mathrm{i}k\alpha \mathbb{I}_{V_0^K}.$$

Using this expression in (31), we obtain (30) and hence

$$\widetilde{B}^K(v_0,\alpha) = \alpha k v_0 + L_1(\alpha)v_0 + g(v_0,\alpha),$$

where $L_1(\alpha)$ is of higher order than $|\alpha|$ and $g(v_0, \alpha)$ is of higher order than $\|v_0\|$. It follows that in the equation

$$0 = \bigl\langle v_0, \widetilde{B}^K(v_0, \alpha)\bigr\rangle = \alpha k \|v_0\|^2 + \bigl\langle v_0, L_1(\alpha)v_0 + g(v_0, \alpha)\bigr\rangle$$

we can factor out $\|v_0\|^2$ and then apply the Implicit Function Theorem to obtain a unique function $\alpha : B_r(0) \to \mathfrak{p}$, for some $r > 0$, near the solution $(0, 0)$.

Using this function we can define a one parameter family of L-equivariant vector fields X_ϵ on S^{2n-1} by

$$X_\epsilon(u) = B^K\bigl(\epsilon u, \alpha(\epsilon)u\bigr).$$

The zeroes of these vector fields correspond to solutions of the bifurcation equation. Since L acts freely on S^{2n-1}, X_ϵ determines a smooth vector field $\widetilde{X}_\epsilon$ on S^{2n-1}/L; the Poincaré–Hopf theorem implies that generically $\widetilde{X}_\epsilon$ has at least

$$\chi\bigl(S^{2n-1}/L\bigr) = \begin{cases} \chi\bigl(\mathbb{CP}^{n-1}\bigr) = n & \text{if } L \simeq S^1, \\ \chi\bigl(\mathbb{CP}^{n-1}/\mathbb{Z}_2\bigr) = n/2 & \text{if } L \simeq S^1 \times \mathbb{Z}_2 \end{cases}$$

equilibria.

The following lemma proves that $X_\epsilon(u)$ is always orthogonal to the tangent space $\mathfrak{l} \cdot u$ of the L-orbit of u, i.e., $\langle X_\epsilon(u), \zeta_{S^{2n-1}}(u)\rangle = 0$ for any $u \in S^{2n-1}$ and $\zeta \in \mathfrak{l}$. Hence the equilibria of $\widetilde{X}_\epsilon$ correspond to orbits of equilibria of X_ϵ, which in turn determine orbits of solutions of the bifurcation equation.

Lemma 5.3. *If* $[\mathrm{Lie}(N_{G_{m_e,\xi}}(K)), \mathfrak{g}_{m_e}^K] = 0$, *then* $\langle X_\epsilon(u), \zeta_{S^{2n-1}}(u)\rangle = 0$ *for any* $u \in S^{2n-1}$ *and* $\zeta \in \mathfrak{l}$.

Proof. We first show that $\widetilde{B}^K(v_0, \alpha)$ is orthogonal to $\mathfrak{l} \cdot v_0$ for any $v_0 \in V_0^K$ and $\alpha \in \mathfrak{p}$.

Given $\alpha \in \mathfrak{p}$, define $\mathcal{H}_\alpha : V_0^K \to \mathbb{R}$ and $\mathbf{j}_\alpha : V_0^K \to \mathfrak{g}^*$ by

$$\mathcal{H}_\alpha(v_0) = \mathcal{H}\bigl(v_0 + v_1(0, v_0, \alpha)\bigr) \quad \text{and} \quad \mathbf{j}_\alpha(v_0) = \mathbf{j}\bigl(v_0 + v_1(0, v_0, \alpha)\bigr).$$

The equivariance of v_1 and triviality of the action on $\mathfrak{p}$ imply that $\mathcal{H}_\alpha$ is $G_{m_e,\xi}$-invariant and $\mathbf{j}_\alpha$ is $G_{m_e,\xi}$-equivariant.

We can choose the space annihilated by V_2 as a complement V_1 to V_0 in V. (If V_2 is identified with V_0 using an inner product, this choice for V_1 is the orthogonal complement to V_0 in V.) In this case,

$$D_N \mathcal{H}^{\Xi(0,v_0,\alpha)}\bigl(0, v_0 + v_1(0, v_0, \alpha)\bigr) \cdot v_1 = 0$$

for any $v_0 \in V_0$, $v_1 \in V_1$, and $\alpha \in \mathfrak{g}_{m_e}$. Hence, given $v_0, u \in V_0^K$, $\alpha \in \mathfrak{p}$, and $\zeta \in \mathfrak{g}_{m_e,\xi} \subset \mathfrak{g}_\mu$, if we set $\eta = \Xi(0, v_0, \alpha)$ and $v = v_0 + v_1(v_0, \alpha)$, then

$$
\begin{aligned}
&\big\langle \widetilde{B}^K(v_0,\alpha), \zeta_{V_0}(v_0)\big\rangle \\
&\quad = D_N\mathcal{H}^\eta(0,v)\zeta_{V_0}(v_0) = D_N\mathcal{H}^\eta(0,v)\big(\big(\mathrm{id} + D_{V_0}v_1(0,v_0,\alpha)\big)\zeta_{V_0}(v_0)\big) \\
&\quad = D\big(\mathcal{H}_\alpha - \mathbf{j}^\alpha_\eta\big)(v_0)\zeta_{V_0}(v_0) = \big\langle \mathrm{ad}^*_\zeta \mathbf{j}_\alpha(v_0), \eta\big\rangle = \big\langle \mathrm{ad}^*_\zeta \mathbf{J}_N(v), \eta\big\rangle.
\end{aligned}
$$

In particular, if $\zeta \in \mathrm{Lie}(N_{G_{m_e,\xi}}(K))$ and $[\mathrm{Lie}(N_{G_{m_e,\xi}}(K)), \mathfrak{g}^K_{m_e}] = 0$, then $\langle \widetilde{B}^K(v_0,\alpha), \zeta_{V_0}(v_0)\rangle = 0$, since $\mathbf{J}_N(v) \in (\mathfrak{g}^*_{m_e})^K$.

To complete the proof, note that the linearity of the action implies that

$$
\begin{aligned}
\big\langle X_\epsilon(u), \zeta_{S^{2n-1}}(u)\big\rangle &= \big\langle \widetilde{B}^K\big(\epsilon u, \alpha(\epsilon u)\big), \zeta_{S^{2n-1}}(u)\big\rangle \\
&= \frac{1}{\epsilon}\big\langle \widetilde{B}^K\big(\epsilon u, \alpha(\epsilon u)\big), \zeta_{V_0}(\epsilon u)\big\rangle = 0. \qquad \square
\end{aligned}
$$

Remark 5.4. Note that Theorem 5.2 provides a (generic) lower bound for the number of branches of critical points bifurcating from m_e. In fact, if m_e has nontrivial isotropy, then in many situations a sheet of critical points bifurcates from m_e, rather than a finite number of one dimensional branches. An example of this phenomenon is given in Section 6. A continuous curve of bifurcation points with nontrivial isotropy appears in many other symmetric Hamiltonian systems, including the Lagrange top and the Riemann ellipsoids. (See, for example, [24–27].) In [24] it is shown that for Lagrangian systems with S^1 symmetry this phenomenon occurs under conditions that are generic within that class of systems.

Remark 5.5. There are two cases in which Theorem 5.2 can be applied in a particularly straightforward manner. First, suppose that the relative equilibrium m_e is such that its momentum value $\mu = \mathbf{J}(m_e)$ has an Abelian isotropy subgroup G_μ. In such situation we automatically have that $[\mathrm{Lie}(N_{G_{m_e,\xi}}(K)), \mathfrak{g}^K_{m_e}] = 0$ for any $K \subset G_{m_e,\xi} \subset G_\mu$ and also, using the techniques introduced in Section 3.2 (see especially Corollary 3.5), the condition on the rigid residual equation can be easily dealt with.

Another case of interest is when m_e is actually an equilibrium with isotropy equal to the entire symmetry group G, i.e., the G-orbit of m_e is m_e itself. Note that in that case $\mathfrak{m} = \mathfrak{q} = \{0\}$ and therefore the rigid residual equation is trivial. Also, the condition $[\mathrm{Lie}(N_{G_{m_e,\xi}}(K)), \mathfrak{g}^K_{m_e}] = 0$ in the statement of the theorem is not necessary in that case since the bifurcation equation is variational (see Remark 3.1) and therefore the associated vector field is orthogonal to the G-orbits, and *a fortiori* to the $N_{G_{m_e,\xi}}(K)$-orbits in V_0^K. It is interesting to note that in this case, the Equivariant Branching Lemma stated in Theorem 5.1 is not applicable, because the parameter η is now missing.

6. An example from wave resonance in mechanical systems

In order to illustrate our method we consider a Hamiltonian system in $\mathbb{R}^8$ (which we identify with $\mathbb{C}^4$), with Hamiltonian function h and symplectic matrix $\mathrm{iI}_{\mathbb{C}^4}$. We will assume that h is invariant under the canonical action of the torus $G = S^1 \times S^1$ on $(\mathbb{C}^4, \mathrm{iI}_{\mathbb{C}^4})$ defined as follows:

$$R_{\phi,\psi}(z_1, z_2, z_3, z_4) = \left(z_1 e^{i\phi}, z_2 e^{i\psi}, z_3 e^{2i\phi}, z_4 e^{2i\psi}\right),$$
$$(\phi, \psi) \in S^1 \times S^1, \quad (z_1, z_2, z_3, z_4) \in \mathbb{C}^4.$$

We assume in addition that the linearized Hamiltonian vector field has two pairs of imaginary eigenvalues, namely, $\pm i\omega$ in the (z_1, z_2) subspace, and $\pm 2i\omega$ in the (z_3, z_4) subspace. This type of $1:2$ resonance occurs in a variety of mechanical systems, such as in capillary-gravity surface waves (see [6] and references therein). There can be additional symmetries in the system, such as reflection symmetry in space (which would act for example by permutation of z_1 with z_2 and of z_3 with z_4) and time reversibility (transforming complex amplitudes to their conjugates). However, assuming these symmetries would not qualitatively affect the subsequent analysis and we shall not consider them in the sequel. In most applications one of the S^1 invariance comes from the transformation of the system into normal form, we refer to [46] and [45] for an extensive bibliography about Hamiltonian normal form theory.

Our goal is the identification of the relative equilibria of the G-equivariant Hamiltonian vector field induced by h. Computations similar to those of [5] show that the general form of a G-invariant, real smooth Hamiltonian h is

$$h = h(X_1, X_2, X_3, X_4, U_1, U_2, V_1, V_2),$$

where

$$X_j = z_j \bar{z}_j, \qquad U_k = \frac{1}{2}\left(z_k^2 \bar{z}_{k+2} + \bar{z}_k^2 z_{k+2}\right),$$
$$V_k = -\frac{i}{2}\left(z_k^2 \bar{z}_{k+2} - \bar{z}_k^2 z_{k+2}\right), \quad k = 1, 2.$$

Moreover, the Lie algebra $\mathfrak{g} \simeq \mathbb{R}^2$ of G acts on $\mathbb{C}^4$ by

$$(\xi_1, \xi_2) \cdot (z_1, z_2, z_3, z_4) \mapsto (i\xi_1 z_1, i\xi_2 z_2, 2i\xi_1 z_3, 2i\xi_2 z_4). \tag{32}$$

The associated momentum map $\mathbf{J}$ can be written as

$$\mathbf{J}(z_1, z_2, z_3, z_4) = \begin{pmatrix} |z_1|^2 + 2|z_3|^2 \\ |z_2|^2 + 2|z_4|^2 \end{pmatrix}. \tag{33}$$

We now write the relative equilibrium equation, $Dh^{\xi}(m) = 0$, in complex coordinates. We set $\xi = (\xi_1, \xi_2)$ and

$$a_j = \frac{\partial h}{\partial X_j}, \qquad b_k = \frac{\partial h}{\partial U_k}, \qquad c_k = \frac{\partial h}{\partial V_k}.$$

Thus

$$D\big(h - \langle \mathbf{J}, \xi\rangle\big)(z) = \big((a_1 - \xi_1)z_1 + (b_1 + \mathrm{i}c_1)\bar{z}_1 z_3, (a_2 - \xi_2)z_2 + (b_2 + \mathrm{i}c_2)\bar{z}_2 z_4, (a_3 - 2\xi_1)z_3 + \frac{1}{2}(b_1 - \mathrm{i}c_1)z_1^2, (a_4 - 2\xi_2)z_4 + \frac{1}{2}(b_2 - \mathrm{i}c_2)z_2^2\big). \tag{34}$$

We can use the symmetries of the system (34) to easily identify a branch of relative equilibria; we will then use the results of the previous sections to find other branches of relative equilibria bifurcating from this branch. The group $H = \mathbb{Z}_2 \times S^1$ is an isotropy subgroup of the G-action on $\mathbb{C}^4$, with fixed-point subspace $\mathrm{Fix}(H) = \{(0, 0, z_3, 0) \mid z_3 \in \mathbb{C}\}$. Therefore, this space is invariant under the map $D(h^\xi)$; specifically,

$$D\big(h - \langle \mathbf{J}, \xi\rangle\big)(0, 0, z_3, 0) = \big(0, 0, (a_3 - 2\xi_1)z_3, 0\big).$$

Here we make the standard identification of $(\mathbb{C}^4)^*$ with $\mathbb{C}^4$. Thus every element of $\mathrm{Fix}(H)$ is a relative equilibrium, each with a one-parameter family of generators $(\hat{\xi}_1, \xi_2)$, where

$$\hat{\xi}_1 = \tfrac{1}{2}a_3(0, 0, X_3, 0, 0, 0) \tag{35}$$

and ξ_2 is arbitrary. The trajectory of each such relative equilibrium is

$$z(t) = \big(0, 0, C\mathrm{e}^{\mathrm{i}\hat{\xi}_1 t + \varphi}, 0\big)$$

and is parameterized by a positive number C and a phase φ. We call this family of relative equilibria RE_I and analyze the bifurcation of new relative equilibria from this family by applying our slice map decomposition in the points $z_e = (0, 0, C, 0)$. Notice that the isotropy subgroup of z_e equals $H := \mathbb{Z}_2 \times S^1$, with Lie algebra is $\mathfrak{h} := \{(0, \alpha)\colon \alpha \in \mathbb{R}\}$.

In constructing a slice mapping using Proposition 2.2, note that the linearity of the phase space $\mathbb{C}^4$ allows us to use the trivial chart map $\psi(u) = z_e + u$, where $u \in \mathbb{C}^4$. The linearization of the momentum map at z_e is

$$D\mathbf{J}(z_e) \cdot (\delta z_1, \delta z_2, \delta z_3, \delta z_4) = \begin{pmatrix} 4C\,\mathrm{Re}(\delta z_3) \\ 0 \end{pmatrix},$$

with $\ker D\mathbf{J}(z_e) = \{(z_1, z_2, \mathrm{i}y, z_4)\colon z_j \in \mathbb{C}, y \in \mathbb{R}\}$. Using the notation introduced in the first sections of the paper we have that the set $\mathfrak{m}$, that is, the orthogonal complement to $\mathfrak{h} = \mathfrak{g}_{z_e}$ in $\mathfrak{g}_\mu = \mathfrak{g}$, and V, the orthogonal complement to $\mathfrak{g} \cdot z_e = \{(0, 0, 2\mathrm{i}\xi C, 0) \mid \xi \in \mathbb{R}\}$ in $\ker D\mathbf{J}(z_e)$, equal

$$\mathfrak{m} \approx \mathfrak{m}^* = \big\{(\eta, 0) \bigm| \eta \in \mathbb{R}\big\} \quad \text{and} \quad V = \big\{(z_1, z_2, 0, z_4) \bigm| z_j \in \mathbb{C}\big\}.$$

Finally, we set $W = \{(0, 0, \eta, 0) \mid \eta \in \mathbb{R}\}$. These choices yield the slice map

$$\Psi(\eta, v) := \big(0, 0, n(\eta), 0\big) + v = \big(z_1, z_2, n(\eta), z_4\big), \quad \text{where } n(\eta) := C + \frac{\eta}{4C}.$$

The composition of the augmented Hamiltonian with this slice map Ψ is

$$\big(\mathcal{H}^{\xi}\big)(\eta, v) = h\big(X_1, X_2, n^2, X_4, nY_1, U_2, nZ_1, V_2\big) - \xi_1\big(X_1 + 2n^2\big) - \xi_2(X_2 + 2X_4),$$

where $Y_1 := \mathrm{Re}(z_1^2)$, $Z_1 := \mathrm{Im}(z_1^2)$, and $n = n(\eta)$.

The analysis of the relative equilibria is simplified by the commutativity of $\mathfrak{g}$, which implies that $\mathfrak{g}_\mu = \mathfrak{g}$ and the two "rigid" equilibrium conditions (RE1) and (RE2) are trivially satisfied. Hence the first nontrivial step in the algorithm is Step 2: The map ω_1 is simply $\omega_1(\eta, \beta, \alpha) = (\hat{\xi}_1 + \beta, \xi_2 + \alpha)$. We set $\tilde{a}_3 = a_3(X_1, X_2, n^2, X_4, nY_1, U_2, nZ_1, V_2) - a_3(0, 0, n^2, 0, 0, 0, 0, 0)$. Then

$$0 = D_{\mathfrak{m}^*}\big(\mathcal{H}^{\omega_1}\big)(\eta, v) = \frac{2\tilde{a}_3 n + b_1 Y_1 + c_1 Z_1 - 4\beta n}{4C}.$$

Solving this equation for β yields

$$\Xi(\eta, v, \alpha) = \big(\hat{\xi}_1 + \beta(\eta, v), \xi_2 + \alpha\big) = \left(\frac{a_3}{2} + \frac{b_1 Y_1 + c_1 Z_1}{4n(\eta)}\xi_2 + \alpha\right)$$

and

$$\begin{aligned}
&D_V\big(\mathcal{H}^{\Xi(\eta,v,\alpha)}\big)(\eta, v) \\
&\quad = \left(\left(a_1 - \frac{a_3}{2} - \frac{b_1 Y_1 + c_1 Z_1}{4n}\right)z_1 + (b_1 + \mathrm{i}c_1)n\bar{z}_1, \big(a_2 - (\xi_2 + \alpha)\big)z_2\right. \\
&\qquad \left. + (b_2 + \mathrm{i}c_2)\bar{z}_2 z_4, \big(a_4 - 2(\xi_2 + \alpha)\big)z_4 + \frac{1}{2}(b_2 - \mathrm{i}c_2)z_2^2\right).
\end{aligned} \tag{36}$$

The bifurcation of relative equilibria from (RE_I) depends on the invertibility of the linearization of the relative equilibrium equation in V at the point $(0, 0)$. The second variation $D_{VV}(\mathcal{H}^{\Xi})(0, 0)$ has eigenvalues and eigenspaces

$$\begin{array}{ll}
\left.\begin{array}{l}
\lambda_1^+ = a_1 - \dfrac{a_3}{2} + C\sqrt{b_1^2 + c_1^2} \quad \text{(simple)} \\
\lambda_1^- = a_1 - \dfrac{a_3}{2} - C\sqrt{b_1^2 + c_1^2} \quad \text{(simple)}
\end{array}\right\}, & V_1 = \big\{(z_1, 0, 0, 0) \,\big|\, z_1 \in \mathbb{C}\big\}, \\
\lambda_2 = a_2 - \xi_2 \quad \text{(double)}, & V_2 = \big\{(0, z_2, 0, 0) \,\big|\, z_2 \in \mathbb{C}\big\}, \\
\lambda_4 = a_4 - 2\xi_2 \quad \text{(double)}, & V_4 = \big\{(0, 0, 0, z_4) \,\big|\, z_4 \in \mathbb{C}\big\}.
\end{array}$$

These eigenvalues depend on C which we can take as a free parameter. Note that the isotypic decomposition of V with respect to the action of H guarantees the decomposition of $D_{VV}(\mathcal{H}^{\Xi})(0, 0)$ into three 2×2 blocks associated to $V_1^+ \oplus V_1^-$, V_2, and V_4, since the action of S^1 separates the z_1 component from z_2 and z_4, while the action of $\mathbb{Z}_2$ separates further the z_2 component from z_4.

There are two kinds of bifurcation points:

(1) *Bifurcation at* λ_1^+ *or* $\lambda_1^- = 0$. As these are simple eigenvalues, the Lyapunov–Schmidt procedure yields one-dimensional bifurcation equations. The conditions of the Hamiltonian Equivariant Branching Lemma are met, hence we can conclude the existence of a bifurcated branch of relative equilibria parameterized by $\eta \in \mathbb{R}$ at each of these points. Note that $\mathbb{Z}_2$ acts as $-\mathrm{Id}$ on the eigenvectors associated with these eigenvalues. Thus it follows that the bifurcation is of *pitchfork* type. The isotropy group of these solutions still contains S^1. Therefore these relative equilibria fill 1-tori, that is, they are still periodic solutions for the Hamiltonian vector field.
Note that in this case, C can be taken as the bifurcation parameter. However this is equivalent to taking η, since W is defined as the subspace $\{(0, 0, C + \eta, 0)\}$ in $\mathbb{C}^4$.
(2) *Bifurcation at* $\lambda_2 = 0$ *or* $\lambda_4 = 0$. In both of these cases, the eigenvalue is double and therefore the space V_0 determined by the Lyapunov–Schmidt procedure is two dimensional. Note that $S^1 \times \{\mathrm{I}\}$ acts trivially on V_2 and V_4. Therefore, the isotropy subgroup is maximal of complex type in both cases. Applying Theorem 5.2 yields at least one branch of circles of relative equilibria in each case. In fact, there is a two-parameter family (modulo symmetry) of relative equilibria containing (RE_I). These solutions live on 2-tori and are quasi-periodic whenever the ratio of the two components of the generator is irrational. What distinguishes these two families, aside from the fact that they bifurcate at different values of ξ_2, is their symmetry: the isotropy of the solutions bifurcating in the z_4 direction is $\mathbb{Z}_2$, while it reduces to the trivial group for those bifurcating in the z_2 direction.

Note that while the bifurcations associated to $\lambda_1^\pm = 0$ generically occur only at isolated values of C, the bifurcations associated to $\lambda_2 = 0$ and $\lambda_4 = 0$ occur for any value of C satisfying the nondegeneracy condition $a_2(0, 0, C^2, 0) \neq a_4(0, 0, C^2, 0)$, since the second component ξ_2 of the generator at z_e can always be chosen to equal $a_2(z_e)$ or $a_4(z_e)$.

We now proceed with the actual solution of the bifurcation equation. We first consider the bifurcation at $\lambda_1^+ = 0$. Generically, the remaining eigenvalues are nonzero at this point; we shall consider only this case. We simplify the algebra by setting $c_1 = 0$. The eigenspace for λ_1^+ is now $V_1^+ = \{(x, 0, 0, 0) \mid x \in \mathbb{R}\}$. Since V_1^+ is invariant under $D_V(\mathcal{H}^{\Xi})$, the uniqueness of v_1 implies that $v_1 \equiv 0$ and the bifurcation equation (B1) is simply $D_V(\mathcal{H}^{\Xi})|_{V_1^+} = 0$, i.e.,

$$0 = D_V\big(\mathcal{H}^{\Xi(\eta,(x_1,0,0),\alpha)}\big)\big(\eta, (x_1, 0, 0)\big) = \big(f_1\big(\eta, x_1^2\big)x_1, 0, 0\big),$$

where

$$\begin{aligned} f_1(\eta, s) := \; & 2a_1\big(s, 0, n^2, 0, ns, 0\big) - a_3\big(s, 0, n^2, 0, ns, 0\big) \\ & - \frac{s}{n} b_1\big(s, 0, n^2, 0, ns, 0\big), \quad n = n(\eta). \end{aligned}$$

Unless we are in the highly degenerate case in which $D_\eta f_1(0, 0) = D_s f_1(0, 0) = 0$, we can use the Implicit Function Theorem to solve for one variable in terms of the other. If, for example, we solve for η as a function of s, we obtain a unique function $\eta : (-\epsilon, \epsilon) \to \mathbb{R}$

for some $\epsilon > 0$ satisfying $f_1(\eta(s), s) = 0$, and hence $D_V(\mathcal{H}^{\Xi(\eta(x_1^2),(x_1,0,0),\alpha)})(\eta(x_1^2), (x_1, 0,0)) = 0$ for all $x_1^2 \in [0, \epsilon)$. Implicit differentiation of $f_2(\eta(s), s) = 0$ yields $\eta(s) = \frac{s}{4C} + o(s^2)$. Note that the group $\{0\} \times S^1$ is an isotropy subgroup of G, with fixed-point space $z_2 = z_4 = 0$. The bifurcation under consideration takes place in this subspace. The case $\lambda_1^- = 0$ is entirely analogous with the eigenspace $V_1^- = \{(\mathrm{i}y, 0, 0, 0) \mid y \in \mathbb{R}\}$.

We now consider the case $\lambda_1^\pm \neq \lambda_2 = 0 \neq \lambda_4$. Again in order to (slightly) simplify the algebra, we set $c_2 = 0$. Application of the Lyapunov–Schmidt procedure yields

$$v_1(\eta, X_2) = \bigl(0, 0, 0, z_4(\eta, X_2)\bigr), \quad \text{where } z_4(\eta, X_2) := \frac{b_2 z_2^2}{2(a_4 - 2(a_2 + \alpha))}.$$

Substituting v_1 into $D_V(\mathcal{H}^{\Xi(\eta,v,\alpha)})(\eta, v)$ yields

$$\begin{aligned} B(\eta, z_2, \alpha) &= D_V\bigl(\mathcal{H} - \mathbf{j}^{\Xi(\eta,z_2+v_1(\eta,X_2),\alpha)}\bigr)\bigl(\eta, z_2 + v_1(\eta, X_2)\bigr) \\ &= \bigl(0, f_2(\eta, X_2, \alpha) z_2, 0, 0\bigr), \end{aligned}$$

where

$$f_2(\eta, X_2, \alpha) := \frac{b_2^2 X_2}{2(a_4 - 2(a_2 + \alpha))} + \alpha;$$

here a_2, a_4, and b_2 are all evaluated at $(0, X_2, n(\eta)^2, 0, 0)$. Since $f_2(0,0,0) = 0$ and $D_\alpha f_2(0,0,0) = 1$, there exists a neighborhood $\mathcal{W}$ of $(0,0)$ in $\mathbb{R} \times [0,\infty)$ and a function $\alpha : \mathcal{W} \to \mathbb{R}$ such that $f_2(\eta, X_2, \alpha(\eta, X_2)) = 0$ for all $(\eta, X_2) \in \mathcal{W}$. Since f_2 depends on z_2 only through $X_2 = |z_2|^2$, each zero of f_2 determines a circle of critical points of $\mathcal{H}^\Xi$. The case $\lambda_4 = 0$ is entirely analogous.

Note that in the cases $\lambda_2 = 0$ and $\lambda_4 = 0$, varying the parameter η simply shifts the real component of z_3, and hence is equivalent to shifting the initial relative equilibrium $z_e = (0, 0, C, 0)$; thus, when computing the complete bifurcation diagram near the line $\{(0, 0, C, 0)\colon\ C \in \mathbb{R}\}$, we find that generically two pitchforks of revolution, one corresponding to $\lambda_2 = 0$ and the other to $\lambda_4 = 0$, emerge from each point $(0, 0, C, 0)$. In addition, there may be isolated points at which conventional (one dimensional) pitchforks emerge, corresponding to $\lambda_1^\pm = 0$.

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