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Hamiltonian Hopf Bifurcation with Symmetry

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Abstract

In this paper we study the appearance of branches of relative periodic orbits in Hamiltonian Hopf bifurcation processes in the presence of compact symmetry groups that do not generically exist in the dissipative framework. The theoretical study is illustrated with several examples.

1. Introduction

Let (V, ω) be a symplectic vector space and G be a compact Lie group acting linearly and symplectically on V. Let $h_\lambda \in C^\infty(V)^G$ be a one-parameter family of G-invariant Hamiltonians such that for each value of the parameter λ, the origin is an equilibrium of the associated Hamiltonian vector field, that is, $\mathbf{d}h_\lambda(0) = 0$ for arbitrary λ. In this paper we will study the nonlinear implications of the following linear behavior: suppose that there is a value of the parameter $\lambda_\circ$ and a pair of eigenvalues $\pm i\,\nu_\circ$ in the spectrum of the linearization at zero of the dynamics induced by the Hamiltonian vector field $X_{h_{\lambda_\circ}}$ that behave as in Fig. 1.1 when we move the parameter λ around $\lambda_\circ$. Such a behavior in the parametrical motion of the eigenvalues is usually referred to as *Hamiltonian Hopf bifurcation* [vdM85], a denomination that we will use here, even though it also appears in the literature as $1 : -1$ resonance, $1 : 1$ non-semisimple resonance, and Krein collision. The reference to the Hopf bifurcation comes from the analogy with the codimension-one non-conservative case in which a one-parameter family of vector fields has a pair of eigenvalues that cross the imaginary axis at a critical value of the parameter (the "classical" Hopf bifurcation). The case of G-equivariant vector fields (G compact) has led to the successful theory of Hopf bifurcation with symmetry which was initiated by [GoS85] and which was described in its most general form in [Fi94] (see also [ChL00] for a comprehensive exposition).

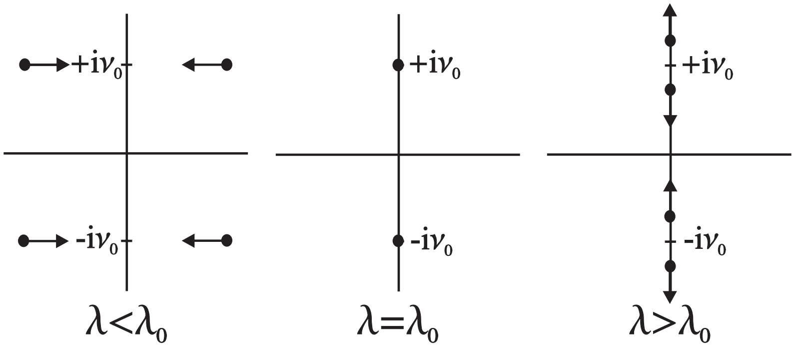

Fig. 1.1. Motion of eigenvalues in a Hamiltonian Hopf bifurcation.

The history of the Hamiltonian Hopf bifurcation in the non-symmetric setup is very long and we shall not attempt to survey it here. We just refer to [MeyS71, Mey86, vdM85, vdM96, Bri90, GMSD95] and references therein for discussions.

The only works that we know of dealing with the Hamiltonian symmetric case are [vdM90], [Vbw90], and [KMS96]. In the first paper it is shown that the non-symmetric results on Hamiltonian Hopf bifurcation can be applied to some of the fixed-point spaces corresponding to isotropy subgroups of the symmetry of the system, provided that certain dimensional restrictions are fulfilled. The second paper [KMS96] studies branches of (stable) three-tori that can be obtained from a Hamiltonian Hopf bifurcation process with a symmetry given by the semi-direct product of D_2 with $T^2 \times S^1$. See also [Bri90a].

Natural dynamical elements that show up in the study of systems that present a continuous symmetry group G are the so called *relative equilibria* (*RE*) and *relative periodic orbits* (*RPOs*), that is, motions that project onto equilibria and periodic orbits in the quotient space V/G, respectively. In our work we will see that whenever a Hopf-like motion of eigenvalues occurs in a Hamiltonian system with symmetry, we can prove the existence of periodic and relative periodic motions at the nonlinear level for values of the parameter close to $\lambda_\circ$. We will estimate the number at each energy level. The existence of periodic motions, in the presence of some dimensional restrictions that we have eliminated, was something known to the authors quoted above. As to the relative periodic orbits, they are found in those papers only after a reduction has been performed that makes the problem equivalent to that of searching for periodic orbits in the relevant quotient space. Since this reduction cannot always be carried out in a straightforward manner, we will follow an approach in which the existence of relative periodic orbits is proved in the original space V. We should point out that, as we will reason later on, the appearance of the relative periodic motions is not compatible with the dimensional restrictions in [vdM90] since in that case only periodic motions can be expected.

Our approach to this problem will be based on the combined use of five tools:

(i) Reduction method of VANDERBAUWHEDE & VAN DER MEER [VvdM95]: it allows us to replace the search for periodic and relative periodic orbits by the search for relative equilibria of a S^1-symmetric associated Hamiltonian system (usually referred to as a *normal form*).

(ii) Generic structure of the generalized eigenspaces corresponding to the colliding eigenvalues [DMM92]: it determines the most plausible reduced space in which we should work after applying (i).

(iii) Equivariant Williamson normal form [MD93]: it is used to normalize the linear term of the equation that defines the S^1-relative equilibria that we are looking for.

(iv) Lyapunov-Schmidt reduction of the finite dimensional equation that defines the S^1-relative equilibria and formulation of the problem in terms of a bifurcation equation of gradient nature [CLOR99] with very specific equivariance properties.

(v) Solution of the bifurcation equation using either topological or analytical methods.

The paper is structured as follows:

- In Section 2 we briefly review the abovementioned tools, set the notation that will be used throughout the paper, and explain in detail the hypotheses under which we will work, along with their implications. The expert can skip this section and use it just as a glossary.
- Section 3 is devoted to Theorem 3.4, which provides a lower estimate on the number of periodic and relative periodic branches that bifurcate from the origin if there is a collision of eigenvalues as in Fig. 1.1.
- In Section 4.1 we study a system of two nonlinearly coupled harmonic oscillators in the presence of a magnetic field, which will lead us to the consideration of the general case of the Hamiltonian Hopf bifurcation in the presence of a $O(2)$ symmetry. We will see that, in contrast to the dissipative case, the $O(2)$-symmetry in a Hamiltonian Hopf bifurcation process gives rise to the appearance of numerous relative periodic motions. This example will also show that, in general, the topological methods utilized in Theorem 3.4 are not powerful enough to detect all the periodic and relative periodic elements of a particular system with a given symmetry, that is, due to the generality of this result one loses sharpness. This circumstance will motivate a more direct approach to the problem in Section 4.3, where we will see that, under additional hypotheses on the group action, sharper general results can be formulated that give account of all the dynamical richness evidenced in the example in Section 4.1.
- In Section 4.4 we use the previous results to show the existence of RPOs in Hamiltonian Hopf phenomena with spherical symmetry.
- For the sake of the clarity in the exposition, the proofs of some of the technical results needed in the main theorem are relegated to an appendix (Section 5) at the end of the paper.

2. Preliminaries and setup

2.1. Hamiltonian symmetric systems

Throughout this paper, our discussions will mostly take place in a symmetric finite-dimensional symplectic vector space (V, ω). The symbol ω denotes the corresponding symplectic form, that is, a closed degenerate two-form on V. In this category, the symmetries of a system are usually encoded in the linear left action of a Lie group G, that we will assume to be compact, via the map $\Phi : G \times V \to V$. We will always deal with *canonical* actions, that is, for any $g \in G$ we have

$$\Phi_g^* \omega = \omega.$$

In this situation, we can associate with any G-invariant function $h \in C^\infty(V)^G$ a G-equivariant vector field $X_h \in \mathfrak{X}(V)^G$ through the equation

$$\mathbf{i}_{X_h}\omega = \mathbf{d}h.$$

The vector field X_h is called the *Hamiltonian vector field* associated with the *Hamiltonian function* h. Every canonical linear action has an associated equivariant *momentum map* $\mathbf{K} : V \longrightarrow \mathfrak{g}^*$ defined by

$$\langle \mathbf{K}(v), \xi \rangle := \frac{1}{2}\omega(\xi \cdot v, v) \quad \text{for any} \quad v \in V,\ \xi \in \mathfrak{g}.$$

In the previous equality, the symbol $\langle \cdot\,, \cdot \rangle$ denotes the natural pairing of the Lie algebra $\mathfrak{g}$ with its dual and $\xi \cdot v$ is the infinitesimal generator at $v \in V$ associated with $\xi \in \mathfrak{g}$, defined by

$$\xi \cdot v = \left.\frac{d}{dt}\right|_{t=0} \exp t\xi \cdot v.$$

Most of the time we will use the notation

$$\mathbf{K}^\xi := \langle \mathbf{K}, \xi \rangle.$$

The momentum map $\mathbf{K}$ is G-equivariant with respect to the previously defined action on V and the coadjoint action of G on $\mathfrak{g}^*$. The importance of the momentum map in our discussion resides in the following two facts:

- *Noether's Theorem*: the level sets of the momentum map are preserved by the Hamiltonian flows associated with any invariant Hamiltonian.
- The *relative equilibria* of a symmetric Hamiltonian system admit the following convenient characterization in terms of the momentum map. Suppose that $v \in V$ is a relative equilibrium of the symmetric Hamiltonian system $(V, \omega, h, G, \mathbf{K} : V \to \mathfrak{g}^*)$, that is, there exists an element $\xi \in \mathfrak{g}$ for which $F_t(v) = \exp t\xi \cdot v$, where F_t is the flow of the vector field X_h. It is easy to check [AM78] that this is equivalent to the point v being a critical point of the *augmented Hamiltonian* defined by $h - \langle \mathbf{K}, \xi \rangle$, that is

$$\mathbf{d}(h - \langle \mathbf{K}, \xi \rangle)(v) = 0.$$

  This characterization will be used extensively.

2.2. Normal form reduction and periodic orbits

In this work we will be interested in the periodic and relative periodic motions around the origin associated with a one-parameter family of G-equivariant Hamiltonian vector fields X_{h_λ}, induced by a family of G-invariant Hamiltonians $h_\lambda \in C^\infty(V)^G$, $\lambda \in \mathbb{R}$, that satisfies the following two hypotheses:

(H1) The Hamiltonians are such that $h_\lambda(0) = 0$ and $\mathbf{d}h_\lambda(0) = 0$ for all λ.
(H2) There is a value $\lambda_\circ$ of the parameter λ for which the G-equivariant infinitesimally symplectic linear map $A_{\lambda_\circ} := D_V X_{h_{\lambda_\circ}}(0)$ is non-singular and has $\pm i\nu_\circ$ in its spectrum ($\nu_\circ \neq 0$).

By the nondegeneracy of the symplectic form ω, hypothesis (H1) is equivalent to the Hamiltonian vector fields X_{h_λ} having an equilibrium at the origin for all λ. The standard way to seek periodic motions around an equilibrium is based on the use of normal forms. We will follow the approach to this tool presented in [VvdM95], which we briefly review in the following paragraphs.

The resonance space. Let (V, ω) be a symplectic vector space. It is easy to show that there is a bijection between linear Hamiltonian vector fields on (V, ω) and quadratic forms on V. Indeed, if $A : V \to V$ is an infinitesimally symplectic linear map, that is, a linear Hamiltonian vector field on (V, ω), its corresponding Hamiltonian function is given by

$$Q_A(v) := \tfrac{1}{2}\omega(Av, v) \qquad \text{for any } v \in V.$$

Also, if A belongs to the symplectic Lie algebra $\mathfrak{sp}(V)$, it admits a unique *Jordan-Chevalley decomposition* [Hu72,VvdM95] of the form $A = A_s + A_n$, where $A_s \in \mathfrak{sp}(V)$ is semisimple (complex diagonalizable), $A_n \in \mathfrak{sp}(V)$ is nilpotent, and the commutator $[A_s, A_n] = 0$. If $i\nu_\circ$ is one of the eigenvalues of $A \in \mathfrak{sp}(V)$ and $T_{\nu_\circ} := \frac{2\pi}{\nu_\circ}$, we define the *resonance space* $U_{\nu_\circ}$ of A with *primitive period* $T_{\nu_\circ}$ as

$$U_{\nu_\circ} := \ker\left(e^{A_s T_{\nu_\circ}} - I\right).$$

The resonance space $U_{\nu_\circ}$ has the following properties (see [Wil36,GoS87] and [VvdM95]):

(i) The space $U_{\nu_\circ}$ is equal to the direct sum of the real generalized eigenspaces of A corresponding to eigenvalues of the form $\pm ik\nu_\circ$, with $k \in \mathbb{N}^*$.
(ii) The pair $(U_{\nu_\circ}, \omega|_{U_{\nu_\circ}})$ is a symplectic subspace of (V, ω).
(iii) The mapping $\theta \in S^1 \mapsto e^{\frac{\theta}{\nu_\circ}A_s}|_{U_{\nu_\circ}}$ generates a symplectic S^1-linear action on $(U_{\nu_\circ}, \omega|_{U_{\nu_\circ}})$, whose associated S^1-invariant momentum map $\mathbf{J} : U_{\nu_\circ} \to \mathrm{Lie}(S^1)^* \simeq \mathbb{R}$ is given by

$$\mathbf{J}(v) = \frac{1}{2\nu_\circ}\omega|_{U_{\nu_\circ}}(A_s v, v).$$

(iv) If (V,ω) is a symplectic representation space of the Lie group G and the Hamiltonian vector field A is G-equivariant (equivalently, the quadratic form Q_A is G-invariant), then the symplectic resonance subspace $(U_{\nu_\circ}, \omega|_{U_{\nu_\circ}})$ is also G-invariant (this follows from the uniqueness of the Jordan-Chevalley decomposition of A, which implies that if A is G-equivariant, so is A_s). Moreover, the S^1 and G actions on $(U_{\nu_\circ}, \omega|_{U_{\nu_\circ}})$ commute, which therefore defines a symplectic linear action of $G \times S^1$ on $U_{\nu_\circ}$. See the Appendix (Section 5) for a sketch of the proof of some of these facts.

The normal form reduction [vdM85,vdM90,VvdM95]. Let (V,ω,h_λ) be a λ-parameter family ($\lambda \in \Lambda$, where Λ is a Banach space) of G-Hamiltonian systems such that $h_{\lambda_\circ}(0)=0$, $\mathbf{d}h_{\lambda_\circ}(0)=0$, and the G-equivariant infinitesimally symplectic linear map $A := DX_{h_{\lambda_\circ}}(0)$ is non-singular and has $\pm i\nu_\circ$ as eigenvalues. Let $(U_{\nu_\circ}, \omega|_{U_{\nu_\circ}})$ be the resonance space of A with primitive period $T_{\nu_\circ}$. For each $k \geqq 0$ there are a C^k mapping $\psi : U_{\nu_\circ} \times \Lambda \to V$ and a C^{k+1} mapping $\hat{h}_\lambda : U_{\nu_\circ} \times \Lambda \to \mathbb{R}$ such that $\psi(0,\lambda)=0$, for all $\lambda \in \Lambda$, $D_{U_{\nu_\circ}}\psi(0,\lambda_\circ) = \mathbb{I}_{U_{\nu_\circ}}$, and $\hat{h}_\lambda$ is a $G \times S^1$-invariant function that coincides with h_λ up to order $k+1$. The interest of normalization is given by the fact that we can prove [VvdM95, Theorem 3.2] that if we stay close enough to zero in $U_{\nu_\circ}$ and to $\lambda_\circ \in \Lambda$, then the S^1-relative equilibria of the $G \times S^1$-invariant Hamiltonian $\hat{h}_\lambda$ are mapped by $\psi(\cdot,\lambda)$ to the set of periodic solutions of (V,ω,h_λ) in a neighborhood of $0 \in V$ with periods close to $T_{\nu_\circ}$. Hence, in our future discussion we will replace the problem of seeking periodic orbits of (V,ω,h_λ) by that of searching for the S^1-relative equilibria of the $G \times S^1$-invariant family of Hamiltonian systems $(U_{\nu_\circ}, \omega|_{U_{\nu_\circ}}, \hat{h}_\lambda)$, which will be referred to as the *equivalent system*. Note that the properties of ψ imply that

$$\mathcal{A} := A|_{U_{\nu_\circ}} = D_{U_{\nu_\circ}} X_{h_{\lambda_\circ}}(0)|_{U_{\nu_\circ}} = D_{U_{\nu_\circ}} X_{h_{\lambda_\circ}|_{U_{\nu_\circ}}}(0) = D_V X_{\hat{h}_{\lambda_\circ}}(0). \tag{2.1}$$

2.3. *Generic structure of the resonance space*

Let (V,ω,h_λ) be the family introduced in the previous section, satisfying hypotheses (H1) and (H2). Let $U_{\nu_\circ}$ be the resonance space associated with the eigenvalues $\pm i\nu_\circ$. After the remarks previously made, we know that this resonance space is a $G \times S^1$-symplectic vector space. The decomposition of $U_{\nu_\circ}$ into $G \times S^1$-irreducible subspaces that can be generically expected when the eigenvalues behave parametrically as in Fig. 1.1 has been studied in [DMM92], where the authors concluded (Proposition 6.1 (3)) that the only generic possibility is

$$U_{\nu_\circ} = U_1 \oplus U_2, \tag{2.2}$$

where U_1 and U_2 are complex dual irreducible subspaces of $U_{\nu_\circ}$ in the sense of [MRS88, Theorem 2.1]. In all that follows we will assume that we are in this generic situation.

Once we know the decomposition (2.2) of $U_{\nu_\circ}$ into irreducibles, we can use the equivariant version of the so called WILLIAMSON normal form [Wil36], due to MELBOURNE & DELLNITZ [MD93], in order to normalize the pair $(\mathcal{A}, \omega|_{U_{\nu_\circ}})$. Table number two in that reference guarantees that there is a basis of the vector space $U_{\nu_\circ}$ in which the simultaneous matrix expressions of $\omega|_{U_{\nu_\circ}}$ and $\mathcal{A}$ are either

(i)

$$\mathcal{A} = \begin{pmatrix} \nu_\circ \mathbb{J}_{2n} & \mathbb{I}_{2n} \\ \mathbf{0} & \nu_\circ \mathbb{J}_{2n} \end{pmatrix} \quad \text{and} \quad \omega|_{U_{\nu_\circ}} = \mathbb{J}_{4n}, \quad \text{or} \tag{2.3}$$

(ii)

$$\mathcal{A} = \begin{pmatrix} \nu_\circ \mathbb{J}_{2n} & \mathbb{I}_{2n} \\ \mathbf{0} & \nu_\circ \mathbb{J}_{2n} \end{pmatrix} \quad \text{and} \quad \omega|_{U_{\nu_\circ}} = -\mathbb{J}_{4n}, \tag{2.4}$$

where $2n = \dim U_1 = \dim U_2$, $\mathbb{I}_{2n}$ is the $2n$-dimensional identity matrix, and $\mathbb{J}_{2n}$ is defined as

$$\mathbb{J}_{2n} = \begin{pmatrix} \mathbf{0} & -\mathbb{I}_n \\ \mathbb{I}_n & \mathbf{0} \end{pmatrix}.$$

Given that the treatment of cases (i) and (ii) is completely analogous, we will focus in all that follows on expression (2.3). Moreover, whenever our family of G-Hamiltonian systems falls into the generic situation described in this paragraph, we will say that it satisfies the condition (H3). For clarity and future reference we state this condition explicitly:

(H3) The resonance space $U_{\nu_\circ}$ corresponding to the eigenvalues $\pm i\nu_\circ$ splits into two complex dual $G \times S^1$-irreducible subspaces. This condition is generic.

2.4. The search for periodic orbits

With the tools that we just introduced we are in position to formulate the equations that characterize the periodic orbits around the origin of the family of Hamiltonian systems that we previously presented. Since these equations will be of much relevance in the search for relative periodic motions that we will carry out in the next section, we will study them in detail.

First of all, and being consistent with the notation previously introduced, let $\mathcal{A} = \mathcal{A}_s + \mathcal{A}_n$ be the Jordan-Chevalley decomposition of $\mathcal{A} \in \mathfrak{sp}_G(U_{\nu_\circ})$. We will denote by $\mathbf{J} : U_{\nu_\circ} \to \mathrm{Lie}(S^1)^* \simeq \mathbb{R}$ the equivariant momentum map associated with the symplectic S^1-linear action defined by $(\theta, v) \mapsto e^{\frac{\theta}{\nu_\circ}\mathcal{A}_s} v$, $\theta \in S^1$, $v \in U_{\nu_\circ}$. Also, for any $\xi \in \mathrm{Lie}(S^1) \simeq \mathbb{R}$ and any $v \in U_{\nu_\circ}$, we will write $\mathbf{J}^\xi(v) := \mathbf{J}(v)\xi$. As we already know, the linearity of the action implies that, for any $\xi \in \mathrm{Lie}(S^1) \simeq \mathbb{R}$ and any $v \in U_{\nu_\circ}$, the momentum map $\mathbf{J}$ is uniquely determined by the expression

$$\mathbf{J}^\xi(v) = \tfrac{1}{2}\omega|_{U_{\nu_\circ}}(\xi \cdot v, v),$$

where the dot in $\xi \cdot v$ means the associated representation of the Lie algebra $\mathrm{Lie}(S^1)$ on $U_{\nu_\circ}$ through the S^1-action. More specifically,

$$\mathbf{J}(v) = \frac{1}{2\nu_\circ}\omega|_{U_{\nu_\circ}}(\mathcal{A}_s v, v).$$

For future reference we note that this relation implies that

$$\mathbf{d}^2\mathbf{J}(0)(v, w) = \omega|_{U_{\nu_\circ}}(\mathcal{A}_s v, w) \qquad \text{for any} \qquad v, w \in U_{\nu_\circ}, \tag{2.5}$$

which in the basis used to write (2.3) admits the following matrix expression:

$$\mathbf{d}^2\mathbf{J}(0) = \begin{pmatrix} \mathbf{0} & \mathbb{J}_{2n} \\ -\mathbb{J}_{2n} & \mathbf{0} \end{pmatrix}. \tag{2.6}$$

As we already said, in the Hamiltonian framework, the search for relative equilibria reduces to the determination of the critical points of the so-called augmented Hamiltonian. In the particular case that we are dealing with, this remark translates into saying that the equivalent system $(U_{\nu_\circ}, \omega|_{U_{\nu_\circ}}, \hat{h}_\lambda)$ has a S^1-relative equilibrium at $v \in U_{\nu_\circ}$ (which represents a periodic orbit of the original system (V, ω, h_λ) with period near $T_{\nu_\circ}$) if and only if there is an element $\xi \in \text{Lie}(S^1)$ for which

$$\mathbf{d}(\hat{h}_\lambda - \mathbf{J}^\xi)(v) = 0. \tag{2.7}$$

Whenever we find a pair (v, ξ) that satisfies (2.7), we will say that v is a relative equilibrium with *velocity* ξ.

Expression (2.7) can be written as a gradient equation, which will be exploited profusely in our subsequent discussion. Indeed, let $\langle \cdot\,, \cdot \rangle$ be a $G \times S^1$-invariant inner product on $U_{\nu_\circ}$ (always available by the compactness of $G \times S^1$). For any $v \in U_{\nu_\circ}$, we define the gradient $\nabla_{U_{\nu_\circ}}(\hat{h}_\lambda - \mathbf{J}^\xi)(v)$ as the unique element in $U_{\nu_\circ}$, such that for $w \in U_{\nu_\circ}$ arbitrary

$$\mathbf{d}(\hat{h}_\lambda - \mathbf{J}^\xi)(v) \cdot w = \langle \nabla_{U_{\nu_\circ}}(\hat{h}_\lambda - \mathbf{J}^\xi)(v), w \rangle.$$

Also for future reference, we recall that the linearization $A_\lambda = D_V X_{h_\lambda}(0)$ of X_{h_λ} at $0 \in V$, is a linear G-equivariant Hamiltonian vector field with associated quadratic Hamiltonian function Q_λ given by

$$Q_\lambda(v) = \tfrac{1}{2}\mathbf{d}^2 h_\lambda(0)(v, v),$$

that is:

$$\mathbf{i}_{A_\lambda}\omega = \mathbf{d}Q_\lambda.$$

The restriction $\mathcal{A}$ of $A_{\lambda_\circ}$ to $U_{\nu_\circ}$ is of course also Hamiltonian but in this case, by (2.1), the associated quadratic Hamiltonian function can be expressed in terms of the Hessian at 0 of the equivalent Hamiltonian $\hat{h}_{\lambda_\circ}$ associated with $h_{\lambda_\circ}$. Indeed,

$$\mathbf{i}_{\mathcal{A}}\omega|_{U_{\nu_\circ}} = \mathbf{d}\mathcal{Q}_{\lambda_\circ}, \tag{2.8}$$

where, for any $v \in U_{\nu_\circ}$,

$$\mathcal{Q}_{\lambda_\circ}(v) = \tfrac{1}{2}\mathbf{d}^2 h_{\lambda_\circ}(0)(v, v) = \tfrac{1}{2}\mathbf{d}^2\hat{h}_{\lambda_\circ}(0)(v, v). \tag{2.9}$$

If we write (2.8) using the basis that produced the canonical form (2.3), the equality (2.9) guarantees that

$$\mathbf{d}^2\hat{h}_{\lambda_\circ}(0) = \mathbf{d}^2 h_{\lambda_\circ}(0)|_{U_{\lambda_\circ}} = -\mathbb{J}_{4n}\mathcal{A} = \begin{pmatrix} \mathbf{0} & \nu_\circ \mathbb{J}_{2n} \\ -\nu_\circ \mathbb{J}_{2n} & -\mathbb{I}_{2n} \end{pmatrix}. \tag{2.10}$$

Invariant splitting of the resonance space $U_{\nu_\circ}$. Using expressions (2.5) and (2.10) we can immediately construct a very convenient splitting of the resonance space $U_{\nu_\circ}$: let $L : U_{\nu_\circ} \to U_{\nu_\circ}$ be the linear map defined by $\langle L(v), w\rangle = \mathbf{d}^2(\hat{h}_{\lambda_\circ} - \mathbf{J}^{\nu_\circ})(0)(v, w)$, for any $v, w \in U_{\nu_\circ}$. Using expressions (2.5), (2.6), and (2.10) we can write, using the basis introduced in (2.3), that we will use in all that follows:

$$L = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbb{I}_{2n} \end{pmatrix}.$$

Since the linear map L is $G \times S^1$-equivariant and self-adjoint we can split $U_{\nu_\circ} = V_0 \oplus V_1$ as the direct sum of the two $G \times S^1$-invariant subspaces,

$$V_0 := \ker L = \left\{ \begin{pmatrix} \mathbf{a} \\ \mathbf{0} \end{pmatrix} \middle| \, \mathbf{a} \in \mathbb{R}^{2n} \right\}, \qquad V_1 := \operatorname{Im} L = \left\{ \begin{pmatrix} \mathbf{0} \\ \mathbf{b} \end{pmatrix} \middle| \, \mathbf{b} \in \mathbb{R}^{2n} \right\}. \tag{2.11}$$

Since by hypothesis (H3) we are in the generic situation, the resonance space $U_{\nu_\circ}$ splits as the sum of two complex dual irreducible subspaces with respect to the $G \times S^1$ representation [DMM92]. Given that by construction V_0 and V_1 are $G \times S^1$-invariant and have the same dimension, the $G \times S^1$ representations on V_0 and V_1 are necessarily complex irreducible. We describe more precisely the interplay between the decomposition $U_{\nu_\circ} = V_0 \oplus V_1$ and the $G \times S^1$ action in the following elementary lemma whose proof is left as an exercise.

Lemma 2.1. *In all the matrix statements below we assume the use of the basis of the canonical form* (2.3)*.*

(a) *Let $g \in G \times S^1$ be arbitrary and $v = v_0 + v_1 \in U_{\nu_\circ}$, with $v_0 \in V_0$ and $v_1 \in V_1$. Then, there exists a orthogonal matrix A_g such that $[A_g, \mathbb{J}_{2n}] = 0$ and*

$$g \cdot v = \begin{pmatrix} A_g & \mathbf{0} \\ \mathbf{0} & A_g \end{pmatrix} \cdot \begin{pmatrix} v_0 \\ v_1 \end{pmatrix}.$$

(b) *The inner product on $U_{\nu_\circ}$ that takes the Euclidean form when expressed in the coordinates corresponding to the basis used to write the canonical form* (2.3) *is $G \times S^1$-invariant.*

Remark 2.2. In all our subsequent discussions we will use the inner product presented in the previous lemma and the basis of the canonical form (2.3).

The quadratic part of the Hamiltonian and a final generic hypothesis. The complex irreducibility of the $G \times S^1$ action on V_0 implies [GSS88, Lemma 3.4] that if $\mathcal{P}_{G\times S^1}(V_0)$ denotes the ring of real $G \times S^1$-invariant polynomials on V_0, we can choose a basis $\{F_1, \ldots, F_l\}$ of $\mathcal{P}_{G\times S^1}(V_0, V_0)$, that is, the finite type $\mathcal{P}_{G\times S^1}(V_0)$ module of $G \times S^1$-equivariant polynomial mappings of V_0 into itself, such that

$$\begin{aligned} F_1 &= \mathbb{I}_{2n}, \\ F_2 &= \mathbb{J}_{2n}, \\ \deg F_k &> 1 \qquad \forall k > 2. \end{aligned} \tag{2.12}$$

Analogously, we can choose a Hilbert basis $\{\theta_1, \ldots, \theta_r\}$ of the module $\mathcal{P}_{G\times S^1}(V_0)$, such that

$$\begin{aligned} \theta_1(v) &= \|v\|^2, \\ \deg\theta_k &> 2 \qquad \forall k > 1. \end{aligned} \tag{2.13}$$

In particular, the $G \times S^1$-invariance of the Hamiltonians $\hat{h}_\lambda$ of the equivalent system $(U_{\nu_\circ}, \omega|_{U_{\nu_\circ}}, \hat{h}_\lambda)$ implies that, for each λ, the second derivative $\mathbf{d}^2\hat{h}_\lambda(0)$, considered as a linear map $\mathbf{d}^2\hat{h}_\lambda(0) : V_0 \oplus V_1 \to V_0 \oplus V_1$ is $G \times S^1$-equivariant. At the same time, since it is a Hessian, it is symmetric and therefore there are functions $\sigma, \rho, \tau, \psi \in C^\infty(\mathbb{R})$ such that:

$$\mathbf{d}^2\hat{h}_\lambda(0) = \begin{pmatrix} \sigma(\lambda)\mathbb{I}_{2n} & \tau(\lambda)\mathbb{I}_{2n} + \psi(\lambda)\mathbb{J}_{2n} \\ \tau(\lambda)\mathbb{I}_{2n} - \psi(\lambda)\mathbb{J}_{2n} & \rho(\lambda)\mathbb{I}_{2n}. \end{pmatrix}, \tag{2.14}$$

where, by (2.10), we have the following initial conditions: $\sigma(\lambda_\circ) = 0$, $\rho(\lambda_\circ) = -1$, $\tau(\lambda_\circ) = 0$, and $\psi(\lambda_\circ) = \nu_\circ$. In all that follows we will assume the following generic hypothesis:

(H4) *Eigenvalues crossing condition*: The one-parameter family of G-Hamiltonian systems (V, ω, h_λ) satisfies the condition $\sigma'(\lambda_\circ) \neq 0$, where $\sigma(\lambda) \in C^\infty(\mathbb{R})$ is the smooth real function introduced in (2.14).

Remark 2.3. The generic hypothesis (H4) is a sufficient (see for instance [DMM92]) but not necessary condition for obtaining a behavior of the eigenvalues as in Fig. 1.1, that is, such an evolution can take place even for systems in which $\sigma'(\lambda_\circ) = 0$.

3. Hamiltonian Hopf bifurcation and relative periodic orbits

The main goal of this section is the statement and proof of a result that will provide an estimate on the number of relative periodic orbits of a one-parameter family of G-Hamiltonian systems (V, ω, h_λ) that satisfies the hypotheses (H1) through (H4), formulated in the previous section.

We will begin by introducing some classical definitions that will make more explicit some of the concepts used in the previous paragraphs.

3.1. Hamiltonian relative periodic orbits

It appears very frequently in examples dealing with symmetric families of Hamiltonian systems that the canonical symmetry group G contains a continuous *globally Hamiltonian symmetry*: suppose that G contains a Lie subgroup H of positive dimension. We say that the canonical action of H on V is globally Hamiltonian when we can associate with it an equivariant momentum map $\mathbf{K} : V \to \mathfrak{h}^*$ which is defined by the fact that its components $\mathbf{K}^\xi := \langle \mathbf{K}, \xi \rangle \in C^\infty(\mathbb{R})$, $\xi \in \mathfrak{h}$, have as associated Hamiltonian vector fields the infinitesimal generators of the action

$$\xi_V(v) = \left.\frac{d}{dt}\right|_{t=0} \exp t\xi \cdot v, \qquad \xi \in \mathfrak{h}, v \in V.$$

Definition 3.1. Let (V, ω, h) be a Hamiltonian system with a symmetry given by the canonical action of the Lie group H on V. The point $v \in V$ is called a *relative periodic point* (*RPP*), if there is a $\tau > 0$ and an element $g \in H$ such that

$$F_{t+\tau}(v) = g \cdot F_t(v) \qquad \text{for any} \qquad t \in \mathbb{R},$$

where F_t is the flow of the Hamiltonian vector field X_h. The set

$$\gamma(v) := \{F_t(v) \mid t > 0\}$$

is called a *relative periodic orbit* (*RPO*) through v. The constant $\tau > 0$ is its *relative period* and the group element $g \in H$ is its *phase shift*.

Proposition 3.2. *Let (V, ω, h) be a Hamiltonian system with a globally Hamiltonian symmetry given by the canonical action of the Lie group H on V with associated momentum map $\mathbf{K} : V \to \mathfrak{h}^*$. If the Hamiltonian vector field $X_{h-\mathbf{K}^\xi}$, $\xi \in \mathfrak{h}$, has a periodic point $v \in V$ with period τ, then the point v is an RPP of X_h with relative period τ and phase shift* $\exp \tau\xi$.

Proof. Let F_t be the flow of the Hamiltonian vector field X_h and $K_t(v) = \exp t\xi \cdot v$ that of $X_{\mathbf{K}^\xi}$. By Noether's Theorem:

$$[X_h, X_{\mathbf{K}^\xi}] = -X_{\{h,\mathbf{K}^\xi\}} = 0,$$

where the bracket $\{\cdot, \cdot\}$ denotes the Poisson bracket associated with the symplectic form ω. Due to this equality, we can write (see for instance [AMR99, Corollary 4.1.27]) the following expression for G_t, the flow of $X_{h-\mathbf{K}^\xi}$:

$$G_t(v) = \lim_{n\to\infty} (F_{t/n} \circ K_{-t/n})^n(v) = (K_{-t} \circ F_t)(v) = \exp -t\xi \cdot F_t(v).$$

Since by hypothesis the point v is periodic for G_t with period τ, we have

$$v = \exp -\tau\xi \cdot F_\tau(v),$$

or, equivalently,

$$F_\tau(v) = \exp \tau\xi \cdot v,$$

as required. □

Remark 3.3. Using the previous proposition, we will reduce the search for RPOs of a generic one-parameter family of G-Hamiltonian systems (V, ω, h_λ) that satisfies conditions (H1)–(H4), to the search for periodic orbits of the vector fields of the form $X_{h_\lambda - \mathbf{K}^\xi}$. The reader should notice that without additional hypotheses on the nature of the $G \times S^1$ action or, more explicitly, on the relation between the G and S^1 actions, it is not possible to be precise about the geometry of the relative periodic solutions that we are going to find. For instance, the G and S^1 actions could coincide, and therefore all that we would obtain out of Proposition 3.2, would be relative equilibria that amount to purely periodic motions (*travelling* or *rotating waves*). All that can be said in general is that once an RPO has been found whose isotropy subgroup with respect to the $G \times S^1$ action is $H \subset G \times S^1$, its trajectory is generically dense in a torus of dimension rank $(N(H)/H)$, where $N(H)$ denotes the normalizer of H in $G \times S^1$.

3.2. The main theorem

Our goal in this section is proving the following result.

Theorem 3.4. *Let* (V, ω, h_λ) *be a one-parameter family of G-Hamiltonian systems that satisfies conditions* (H1)–(H4). *Suppose that G contains a Lie subgroup H of positive dimension with associated equivariant momentum map* $\mathbf{K} : V \to \mathfrak{h}^*$. *Let* $U_{\nu_\circ}$ *be the resonance space with primitive period* $T_{\nu_\circ}$. *Then, for each* $\xi \in \mathfrak{h}$ *whose norm* $\|\xi\|$ *is small enough, there are at least, in each energy level close to zero and for each value of the parameter* λ *near* $\lambda_\circ$, *as many relative periodic orbits as the number of equilibria of an arbitrary* $G^\xi \times S^1$*-equivariant vector field defined on the unit sphere on* V_0. *The symbol* G^ξ *denotes the adjoint isotropy subgroup of the element* $\xi \in \mathfrak{h}$, *that is,*

$$G^\xi = \{g \in G \mid \mathrm{Ad}_g \xi = \xi\}.$$

Remark 3.5. If we are just interested in looking for purely periodic orbits it suffices to use Theorem 3.4 with $\xi = 0$. Conversely, if we use this result with a value of the parameter $\xi \neq 0$ we cannot conclude that the predicted RPOs are not trivial, that is, that they are not just periodic orbits. This point will become much clearer in the examples presented in the following sections.

Remark 3.6. In terms of practical applications, the relevance of Theorem 3.4 is given by the fact that the estimate that it provides in terms of the number of equilibria of an equivariant vector field on the sphere can sometimes be calculated via topological arguments, as we will see later on.

Proof. We will work in the basis of the resonance space $U_{\nu_\circ}$ provided by the equivariant Williamson normal form, in particular we will use the matrix expressions (2.3), which are consistent with the decomposition $U_{\nu_\circ} = V_0 \oplus V_1$ presented in (2.11). Recall that the subspaces V_0 and V_1 are $G \times S^1$-invariant. Abusing the notation a little, we will use the symbol ξ to denote both an element of the Lie algebra $\mathfrak{h} \subset \mathfrak{g}$ and its representation on V_0 and V_1. Using Lemma 2.1 we can write, for each $v = v_0 + v_1 \in U_{\nu_\circ}$ represented in the previously mentioned basis,

$$\xi_{U_{\nu_\circ}}(v) = \xi \cdot v_0 + \xi \cdot v_1 = \begin{pmatrix} \xi & \mathbf{0} \\ \mathbf{0} & \xi \end{pmatrix} \begin{pmatrix} v_0 \\ v_1 \end{pmatrix}.$$

Note that, also by Lemma 2.1, the matrix ξ is skew-symmetric, $\xi^T = -\xi$, therefore normal, and hence diagonalizable. The same Lemma implies that the linear map $\xi : V_0 \to V_0$ associated with $\xi \in \mathfrak{h}$ commutes with $\mathbb{J}_{2n}$, $[\xi, \mathbb{J}_{2n}] = 0$, and consequently these two endomorphisms can be simultaneously diagonalized.

We recall that,

$$\langle \mathbf{K}(v), \xi \rangle = \frac{1}{2}\omega(\xi \cdot v, v) = \frac{1}{2}(v_0, v_1) \begin{pmatrix} \mathbf{0} & \xi \\ -\xi & \mathbf{0} \end{pmatrix} \begin{pmatrix} v_0 \\ v_1 \end{pmatrix}.$$

In particular,

$$\mathbf{d}^2\mathbf{K}^\xi(0) = \begin{pmatrix} \mathbf{0} & \xi \\ -\xi & \mathbf{0} \end{pmatrix}.$$

We start the proof by defining the $\mathbb{R} \times \mathfrak{h}$-parameter family of Hamiltonian functions given by

$$h_{\lambda,\xi} = h_\lambda - \mathbf{K}^\xi.$$

Due to the hypotheses on the family h_λ, the quadratic nature of the momentum map $\mathbf{K}$, and the fact that $h_{\lambda,0} = h_\lambda$, the family $h_{\lambda,\xi}$ satisfies the hypotheses of the Normal Form Reduction Theorem [VvdM95]. Therefore, a new family $\hat{h}_{\lambda,\xi}$ can be constructed such that, for any value (λ, ξ) of the parameters, the Hamiltonian $\hat{h}_{\lambda,\xi}$ is S^1-invariant with respect to the action generated by the semisimple part of the linearization at zero of $X_{h_{\lambda_\circ,0}} = X_{h_{\lambda_\circ}}$, that is, $(\theta, v) \mapsto \mathrm{e}^{\frac{\theta}{\nu_\circ}\mathcal{A}^s} v$, $\theta \in S^1$, with

$$\mathcal{A}^s = \begin{pmatrix} \nu_\circ \mathbb{J}_{2n} & \mathbf{0} \\ \mathbf{0} & \nu_\circ \mathbb{J}_{2n} \end{pmatrix}. \tag{3.1}$$

The Normal Form Reduction Theorem guarantees that the S^1-relative equilibria of $\hat{h}_{\lambda,\xi}$ are in correspondence with the periodic orbits $h_{\lambda,\xi}$ which, by Proposition 3.2, are RPOs of h_λ. The quadratic nature of the momentum map $\mathbf{K}$ and its S^1-invariance imply that $\hat{h}_{\lambda,\xi}$ can be chosen to be of the form

$$\hat{h}_{\lambda,\xi} = \hat{h}_\lambda - \mathbf{K}^\xi,$$

with $\hat{h}_\lambda$ the normal form for the family h_λ.

As a result of these premises, the RPOs that we are looking for will be given by the critical points of the function $\hat{h}_\lambda - \mathbf{K}^\xi - \mathbf{J}^{\zeta+\alpha}$, that is, the elements $(v, \alpha, \lambda, \xi) \in U_{\nu_\circ} \times \mathbb{R} \times \mathbb{R} \times \mathfrak{h}$ for which the function

$$F^\zeta(v, \alpha, \lambda) := \nabla_{U_{\nu_\circ}} \left(\hat{h}_\lambda - \mathbf{K}^\xi - \mathbf{J}^{\zeta+\alpha} \right)(v) \tag{3.2}$$

has a zero. As customary, the gradient in the previous expression is constructed using the inner product introduced in Lemma 2.1.

Lyapunov-Schmidt reduction and the bifurcation equation The linearization $L^\zeta : U_{\nu_\circ} \to U_{\nu_\circ}$ of (3.2) at the point $(0, 0, \lambda_\circ, 0)$ produces, in the usual basis, the expression:

$$L^\zeta = \mathbf{d}^2\left(\hat{h}_\lambda - \mathbf{J}^\zeta\right)(0) = \mathbf{d}^2\left(h_\lambda - \mathbf{J}^\zeta\right)(0)$$
$$= \begin{pmatrix} \mathbf{0} & (1 - \frac{\zeta}{\nu_\circ})\nu_\circ \mathbb{J}_{2n} \\ -(1 - \frac{\zeta}{\nu_\circ})\nu_\circ \mathbb{J}_{2n} & -\mathbb{I}_{2n} \end{pmatrix}. \tag{3.3}$$

By looking at this matrix expression we see that it is possible to Lyapunov-Schmidt reduce the bifurcation problem posed in (3.2) whenever $\zeta = \nu_\circ$, which we will assume in what follows. In those circumstances $\ker L^{\nu_\circ} = V_0$, $\operatorname{Im} L^{\nu_\circ} = V_1$. Let $\mathbb{P} : U_{\nu_\circ} \to V_0$ be the $G \times S^1$-equivariant projection associated with the splitting $U_{\nu_\circ} = V_0 \oplus V_1$. The equation $(\mathbb{I} - \mathbb{P})F^{\nu_\circ}(v_0 + v_1, \alpha, \lambda, \xi) = (\mathbb{I} - \mathbb{P})\nabla_{U_{\nu_\circ}}\left(\hat{h}_\lambda - \mathbf{K}^\xi - \mathbf{J}^{\nu_\circ+\alpha}\right)(v_0 + v_1) = 0$ defines, via the Implicit Function Theorem, a function $v_1 : V_0 \times \mathbb{R} \times \mathbb{R} \times \mathfrak{h} \to V_1$, such that

$$\begin{aligned}(\mathbb{I} - \mathbb{P})&F^{\nu_\circ}(v_0 + v_1(v_0, \alpha, \lambda, \xi), \alpha, \lambda, \xi)\\ &= (\mathbb{I} - \mathbb{P})\nabla_{U_{\nu_\circ}}\left(\hat{h}_\lambda - \mathbf{K}^\xi - \mathbf{J}^{\nu_\circ+\alpha}\right)(v_0 + v_1(v_0, \alpha, \lambda, \xi)) = 0.\end{aligned} \tag{3.4}$$

Notice that the function $v_1(v_0, \alpha, \lambda, \xi)$ is $G^\xi \times S^1$-equivariant, since this is the symmetry under which $F^{\nu_\circ}$ is equivariant, that is, for any $g \in G^\xi \times S^1$, we have $v_1(g \cdot v_0, \alpha, \lambda, \xi) = g \cdot v_1(v_0, \alpha, \lambda, \xi)$.

The final Lyapunov-Schmidt $G^\xi \times S^1$-equivariant *reduced bifurcation equation*, whose zeros provide us with the RPOs that we are after, is given by $B : V_0 \times \mathbb{R} \times \mathbb{R} \times \mathfrak{h} \to V_0$, where

$$\begin{aligned}&B(v_0, \alpha, \lambda, \xi)\\ &\quad= \mathbb{P}F^{\nu_\circ}(v_0 + v_1(v_0, \alpha, \lambda, \xi), \alpha, \lambda, \xi)\\ &\quad= \mathbb{P}\nabla_{U_{\nu_\circ}}\left(\hat{h}_\lambda - \mathbf{K}^\xi - \mathbf{J}^{\nu_\circ+\alpha}\right)(v_0 + v_1(v_0, \alpha, \lambda, \xi))\\ &\quad= \nabla_{U_{\nu_\circ}}\left(\hat{h}_\lambda - \mathbf{K}^\xi - \mathbf{J}^{\nu_\circ+\alpha}\right)(v_0 + v_1(v_0, \alpha, \lambda, \xi)) \quad \text{(by (3.4))}.\end{aligned} \tag{3.5}$$

We collect the main properties of the reduced bifurcation equation in the following

Lemma 3.7. *The reduced bifurcation equation* (3.5) *is $G^\xi \times S^1$-equivariant with respect to the action of this Lie group on V_0 and it is the gradient of a $G^\xi \times S^1$-invariant function defined on V_0, that is,*

$$B(v_0, \alpha, \lambda, \xi) = \nabla_{V_0} g(v_0, \alpha, \lambda, \xi),$$

where the function $g : V_0 \times \operatorname{Lie}(S^1) \times \mathbb{R} \times \mathfrak{h} \to V_0$ is defined by

$$g(v_0, \alpha, \lambda, \xi) = (\hat{h}_\lambda - \mathbf{J}^{\nu_\circ+\alpha} - \mathbf{K}^\xi)(v_0 + v_1(v_0, \alpha, \lambda, \xi)).$$

Proof. The $G^\xi \times S^1$ equivariance is a direct consequence of the construction of B. As to the gradient character of B, note first that for any $w \in V_1$ we have

$$\begin{aligned}\langle F^{\nu_\circ}&(v_0 + v_1(v_0, \alpha, \lambda, \xi), \alpha, \lambda, \xi), w\rangle \\ &= \langle F^{\nu_\circ}(v_0 + v_1(v_0, \alpha, \lambda, \xi), \alpha, \lambda, \xi), (\mathbb{I} - \mathbb{P})w\rangle \\ &= \langle (\mathbb{I} - \mathbb{P})F^{\nu_\circ}(v_0 + v_1(v_0, \alpha, \lambda, \xi), \alpha, \lambda, \xi), w\rangle = 0, \end{aligned} \tag{3.6}$$

where the last equality follows from the construction of the function v_1 through expression (3.4). Now, let $u \in V_0$ arbitrary. We write:

$$\begin{aligned}&\langle B(v_0, \alpha, \lambda, \xi), u\rangle \\ &\quad = \langle \mathbb{P}F^{\nu_\circ}(v_0 + v_1(v_0, \alpha, \lambda, \xi), \alpha, \lambda, \xi), u\rangle \\ &\quad = \langle F^{\nu_\circ}(v_0 + v_1(v_0, \alpha, \lambda, \xi), \alpha, \lambda, \xi), u\rangle \\ &\quad = \langle F^{\nu_\circ}(v_0 + v_1(v_0, \alpha, \lambda, \xi), \alpha, \lambda, \xi), u + D_{V_0}v_1(v_0, \alpha, \lambda, \xi)\cdot u\rangle \quad \text{(by (3.6))} \\ &\quad = \langle \nabla_{U_{\nu_\circ}}(\hat{h}_\lambda - \mathbf{J}^{\nu_\circ+\alpha} - \mathbf{K}^\xi)(v_0 + v_1(v_0, \alpha, \lambda, \xi)), u + D_{V_0}v_1(v_0, \alpha, \lambda, \xi)\cdot u\rangle \\ &\quad = \mathbf{d}(\hat{h}_\lambda - \mathbf{J}^{\nu_\circ+\alpha} - \mathbf{K}^\xi)(v_0 + v_1(v_0, \alpha, \lambda, \xi))\cdot(u + D_{V_0}v_1(v_0, \alpha, \lambda, \xi)\cdot u) \\ &\quad = \mathbf{d}g(v_0, \alpha, \lambda, \xi)\cdot u = \langle \nabla_{V_0}g(v_0, \alpha, \lambda, \xi), u\rangle, \end{aligned}$$

as required. This construction is a particular case of the one carried out in [GMSD95] and [CLOR99]. □

Notational simplification. In order to make notation a little bit simpler we will assume in the rest of the proof, without loss of generality, that the system has been scaled in such a way that $\nu_\circ = 1$ and $\lambda_\circ = 0$.

The following lemmas provide a local description of the reduced bifurcation equation that will be much needed.

Lemma 3.8. *The function v_1 introduced in* (3.4) *has the following two properties:*

(i)

$$v_1(0, \alpha, \lambda, \xi) = 0 \qquad \textit{for all} \qquad \alpha, \lambda \in \mathbb{R}, \ \textit{and}\ \xi \in \mathfrak{h}. \tag{3.7}$$

(ii)

$$D_{V_0}v_1(0, \alpha, \lambda, \xi) = -\frac{\tau(\lambda)}{\rho(\lambda)}\mathbb{I}_{2n} - \frac{(1+\alpha) - \psi(\lambda)}{\rho(\lambda)}\mathbb{J}_{2n} - \frac{1}{\rho(\lambda)}\xi. \tag{3.8}$$

Proof. Part (i) is a consequence of the uniqueness of the solutions provided by the Implicit Function Theorem. The proof of part (ii) is supplied in the Appendix, Section 5.2. □

The proof of the following lemma is a lengthy but straightforward computation.

Lemma 3.9. *Let $B(v_0, \alpha, \lambda, \xi)$ be the reduced bifurcation equation, then:*

(i)

$$D_{V_0}B(0,\alpha,\lambda,\xi) = \frac{\sigma(\lambda)\rho(\lambda) - \tau^2(\lambda) - ((1+\alpha) - \psi(\lambda))^2}{\rho(\lambda)}\mathbb{I}_{2n} + \frac{2\left[(1+\alpha) - \psi(\lambda)\right]}{\rho(\lambda)}\mathbb{J}_{2n}\xi + \frac{\xi^2}{\rho(\lambda)}.$$

(ii) *The principal part of the reduced bifurcation equation is given by the expression:*

$$B(v_0,\alpha,\lambda,\xi) = (\lambda\sigma'(0) + \alpha^2)v_0 - \xi^2 v_0 - 2\alpha\mathbb{J}_{2n}\xi v_0 - 2\psi'(0)\alpha\lambda v_0 + 2\psi'(0)\lambda\mathbb{J}_{2n}\xi v_0 + C\left(v_0^{(3)}\right) + h.o.t., \tag{3.9}$$

where $C\left(v_0^{(3)}\right)$ *is the trilinear operator obtained by taking the gradient of the fourth order term in the* v_0*-expansion of* $\hat{h}_{\lambda_\circ}(v_0 + v_1(v_0,0,0,0))$.

We now write the reduced bifurcation equation in polar coordinates, that is, we define

$$B_p(r,u_0,\alpha,\lambda,\xi) = B(ru_0,\alpha,\lambda,\xi),$$

where $r \in \mathbb{R}$ and $u_0 \in S^{\dim V_0 - 1}$. We introduce the function

$$F(r,u_0,\alpha,\lambda,\xi) = \frac{\langle B(ru_0,\alpha,\lambda,\xi), u_0\rangle}{r}.$$

By looking at (3.9) it is clear that the function F is smooth at the origin, $F(0,0,0,0,0) = 0$ and that $D_\lambda F(0,0,0,0,0) = \sigma'(0) \neq 0$, by hypothesis (H4). Therefore, the Implicit Function Theorem guarantees the existence of a smooth function $\lambda(r,u_0,\alpha,\xi)$ such that $\lambda(0,0,0,0) = 0$ and $F(r,u_0,\alpha,\lambda(r,u_0,\alpha,\xi),\xi) = 0$. This equality implies that if we substitute the function $\lambda(r,u_0,\alpha,\xi)$ on the reduced bifurcation equation, this time considered as a vector field on V_0, we obtain a new (α,ξ)-parameter dependent vector field

$$G(r,u_0,\alpha,\xi) = B_b(r,u_0,\alpha,\lambda(r,u_0,\alpha,\xi),\xi) \tag{3.10}$$

which due to the fact that $\langle B_b(r,u_0,\alpha,\lambda(r,u_0,\alpha,\xi),\xi),u_0\rangle = 0$ is, for each small enough fixed value of r, a $G^\xi \times S^1$-equivariant vector field on the sphere on V_0 of radius r, whose zeros constitute solutions of the reduced bifurcation equation. □

3.3. Method for the optimal use of Theorem 3.4

The optimal and most organized way to apply Theorem 3.4 consists of using the estimate it provides in the fixed-point subspaces V_0^H corresponding to the various subgroups H in the lattice of isotropy subgroups of the $G \times S^1$ action on V_0, replacing the group $G \times S^1$ by $N(H)$, which is a group that acts on V_0^H (not necessarily in an irreducible manner). The symbol $N(H)$ denotes the normalizer of H in $G \times S^1$ and V_0^H is the vector subspace of V_0 formed by the vectors fixed by H. We make this comment more explicit in the following paragraphs.

Let H be a subgroup of $G \times S^1$. If $\pi : G \times S^1 \to G$ denotes the canonical projection and $\pi(H) =: K \subset G$, Proposition 7.2 in [GSS88] guarantees the existence of a group homomorphism $\theta : K \to S^1$ such that

$$H = \{(k, \theta(k)) \in G \times S^1 \mid k \in K\}. \tag{3.11}$$

In our discussion we will be concerned with *spatiotemporal symmetries*, that is, subgroups H of $G \times S^1$ for which the homomorphism $\theta : K \to S^1$ is nontrivial. Using the characterization (3.11) it is straightforward to see that

$$N(H) = N_G(K) \times S^1.$$

The $N_G(K)$ action on $U^H_{\nu_\circ}$ is globally Hamiltonian with momentum map $\mathbf{K}^H : U^H_{\nu_\circ} \to \mathrm{Lie}\,(N_G(K))^*$ given by the restriction of the G-momentum map to $U^H_{\nu_\circ}$, that is, for any $v \in U^H_{\nu_\circ}$ and any $\xi \in \mathrm{Lie}\,(N_G(K))$, we have that

$$\langle \mathbf{K}^H(v), \xi \rangle = \langle \mathbf{K}(v), \xi \rangle.$$

The same statement applies to the S^1 action. Using these objects we can reformulate Theorem 3.4 on the fixed point spaces V_0^H.

Corollary 3.10. *Let* (V, ω, h_λ) *be a one-parameter family of G-Hamiltonian systems that satisfies conditions* (H1)–(H4). *Let* H *be a spatiotemporal isotropy subgroup of the* $G \times S^1$ *action on* V_0*, such that* $\dim V_0^H = 2k$ *and* $K := \pi(H)$*. Then, for each* $\xi \in \mathrm{Lie}\,(N_G(K))$ *whose norm* $\|\xi\|$ *is small enough, there are at least in each energy level close to zero and for each value of the parameter* λ *near* $\lambda_\circ$*, as many relative periodic orbits as the number of equilibria of an arbitrary* $N_G(K)^\xi \times S^1$*-equivariant vector field on the unit sphere on* V_0^H*. The relative periods of these RPOs are close to* $T_{\nu_\circ}$*, and their phase shifts are close to* $\exp T_{\nu_\circ}\xi$*. The symbol* $N_G(K)^\xi$ *denotes the adjoint isotropy subgroup of the element* $\xi \in \mathrm{Lie}\,(N_G(K))$*, that is,*

$$N_G(K)^\xi = \{g \in N_G(K) \mid \mathrm{Ad}_g \xi = \xi\}.$$

The mapping $\pi : G \times S^1 \to G$ *denotes the canonical projection.*

3.4. Periodic orbits with maximal isotropy subgroup

As we already said, both the previous result and Theorem 3.4 can be used to look for purely periodic motions by taking in their respective statements $\xi = 0$. A situation of special interest takes place when H is a maximal isotropy subgroup of the $G \times S^1$ action on V_0. In the presence of maximality we have at our disposal the following convenient result:

Lemma 3.11. *Let* H *be a maximal isotropy subgroup of the compact* $G \times S^1$ *action on* V_0*. Let* N *be the Lie group* $N(H)/H$ *and* N^0 *be the connected component of the identity of* N*. Then either*

(i) $N^0 \simeq S^1$, *and* $N/N^0 = \{Id\}$ *or* $N/N^0 \simeq \mathbb{Z}_2$, *or*
(ii) $N^0 \simeq SU(2)$ *and* $N \simeq SU(2)$.

In the first case we say that H is a maximal complex subgroup. In the second case we say that H a maximal quaternionic subgroup.

Proof. It is a straightforward combination of the general result for linear actions of compact Lie groups [Bre72, G83, GSS88] with Proposition 12.5 in [GoSt85] that eliminates the possibility of having real maximal isotropy subgroups when the compact group in question is $G \times S^1$. □

Using the previous lemma and an additional genericity hypothesis, the estimate given in Theorem 3.4 can be made very explicit:

Corollary 3.12. *Let $(V,\ \omega,\ h_\lambda)$ be a generic one-parameter family of G-Hamiltonian systems that satisfies conditions* (H1)–(H4). *Let H be a maximal isotropy subgroup of the $G \times S^1$ action on V_0 such that* $\dim(V_0^H) = l \neq 0$. *Then:*

(i) *If $N^0 \simeq S^1$ there are at least $l/2$ (if $N/N^0 = \{Id\}$) or $l/4$ (if $N/N^0 \simeq \mathbb{Z}_2$) branches of periodic solutions with isotropy H coming out of the origin, with periods close to $T_{\nu_\circ}$, as the parameter λ is varied.*
(ii) *If $N^0 \simeq SU(2)$ there are at least $l/4$ branches of periodic solutions with isotropy H coming out of the origin, with periods close to $T_{\nu_\circ}$, as the parameter λ is varied.*

Proof. We will adapt to our problem the approach followed in [CKM95, Koe95] for rotating waves. The main idea behind the proof consists of using the maximality hypothesis to give a numerical evaluation of the estimate in Corollary 3.10, that is, the number of equilibria of a $N(H)/H$-equivariant vector field on the sphere S^{l-1}.

More specifically, let $G^H := G|_{V_0^H}$ be the restriction of the vector field G on V_0, defined in (3.10), to the fixed point set V_0^H, and $G_r^H(u_0, \alpha) := G^H(r, u_0, \alpha)$ be the N-equivariant vector field on S_r^{l-1} obtained by fixing r in the mapping G^H (note that in our case $\xi = 0$ since we are looking for periodic orbits). The zeros of this vector field are in one-to-one correspondence with the solutions that we search. Due to the maximality hypothesis on the subgroup H, the N-action on the sphere S_r^{l-1} is free and therefore the corresponding orbit space S_r^{l-1}/N is a smooth manifold onto which we can project the N-equivariant vector field G_r^H. Let $\bar{G}_r^H$ be the projected vector field. Due to the genericity hypothesis in the statement, the Poincaré-Hopf Theorem allows us to say that $\bar{G}_r^H$ has at least $\chi(S_r^{l-1}/N)$ equilibria, where χ denotes the Euler characteristic.Now recall that B is a gradient (see Lemma 3.7) and consequently so is its restriction to V_0^H. Therefore, these equilibria correspond to equilibria of the restriction of the reduced bifurcation equation to V_0^H.

In order to conclude our argument it is enough to show that $\chi(S^{l-1}/N)$ corresponds to the estimates provided in the statement of the theorem. In the first case, when $N^0 \simeq S^1$, the dimension of V_0^H is necessarily even (we will write $l = 2k$ for certain $k \in \mathbb{N}$) and there are two possibilities: the quotient N/N^0 is either $\{Id\}$ or it is isomorphic to $\mathbb{Z}_2$. If $N/N^0 = \{Id\}$:

$$\chi(S^{l-1}/N) = \chi((S^{l-1}/N^0)/(N/N^0)) = \chi(S^{2k-1}/S^1) = \chi(\mathbb{CP}^{k-1}) = k = \tfrac{1}{2}l.$$

If $N/N^0 \simeq \mathbb{Z}_2$:

$$\chi(S^{l-1}/N) = \chi((S^{l-1}/N^0)/(N/N^0)) = \chi(\mathbb{CP}^{k-1}/\mathbb{Z}_2) = \tfrac{1}{2}k = \tfrac{1}{4}l,$$

where we used the well-known fact that if G is a finite group acting freely on a manifold M, then (see for instance [Kaw91, Corollary 5.22])

$$\chi\left(\frac{M}{G}\right) = \frac{\chi(M)}{|G|}.$$

Finally, if H is maximal quaternionic, then $l = \dim(V_0^H) = 4k$ for some $k \in \mathbb{N}$, necessarily, and

$$\chi(S^{l-1}/N) = \chi(S^{4k-1}/SU(2)) = \chi(\mathbb{HP}^{k-1}) = k = \tfrac{1}{4}l.$$

The calculation of the Euler characteristic $\chi(\mathbb{HP}^{k-1})$ of the quaternionic projective space is made using an argument based the spectral series of Leray (see for instance [BT82]).

The computations that we just carried out give us periodic orbits for a fixed r. Moving smoothly this parameter we obtain the branches required in the statement of the theorem. □

Remark 3.13. The theorem by van der Meer [vdM90] on the Hamiltonian equivariant Hopf bifurcation can be interpreted in the context of the previous corollary. Indeed, this result states the existence of branches of periodic orbits when the dimension of $V_0^H = 2$. This hypothesis puts us in case (i) of Corollary 3.12 because it implies that the isotropy subgroup is maximal and moreover that $N^0 \simeq S^1$ (see Lemma 3.11).

4. Bifurcation of non-periodic relative periodic orbits in the presence of extra hypotheses

The tools presented in Theorem 3.4 for the search of RPOs based on topological methods produce estimates that, as we will see in the following examples, have some limitations; in particular we have no example where it guarantees the bifurcation of non-periodic RPOs. This circumstance has motivated us to use a more analytical approach under dimensional hypotheses that are satisfied in very relevant situations. A detailed study of the bifurcation equation in the presence of these hypotheses will provide us with sharper estimates that completely describe all the bifurcation phenomena that we see in the examples.

4.1. Motivating example: two coupled harmonic oscillators subjected to a magnetic field

We consider the system formed by two identical particles with unit charge and mass m in the XY-plane, subjected to identical repulsive harmonic forces, to a homogeneous magnetic field perpendicular in direction to the plane of motion XY, and to an interaction potential that depends only on the positions and that will preserve a certain group of symmetries. We will denote by (q_1, q_2) the coordinates of the configuration space of the first particle and by (q_3, q_4) those of the second one. If the magnetic field is induced by the vector potential

$$\mathbf{A}(x, y, z) = \gamma(-y, x, 0),$$

the Lagrangian function associated with the system is

$$\begin{aligned} L(\mathbf{q}, \dot{\mathbf{q}}) = {} & \tfrac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2 + \dot{q}_4^2) + \tfrac{1}{2}k(q_1^2 + q_2^2 + q_3^2 + q_4^2) \\ & + \gamma(q_1\dot{q}_2 - q_2\dot{q}_1) + \gamma(q_3\dot{q}_4 - q_4\dot{q}_3) - f(\pi_1^i, \pi_2^i, \pi_3^i), \end{aligned} \tag{4.1}$$

where

$$\begin{aligned} &\pi_1^i = q_i^2 + q_{i+2}^2, \quad \pi_2^i = p_i^2 + p_{i+2}^2, \\ &\pi_3^i = p_i q_{i+2} - p_{i+2} q_i, \quad \pi_4^i = q_i p_i + q_{i+2} p_{i+2}, \quad i \in \{1, 2\}, \end{aligned}$$

f is a higher order function on its variables that expresses a nonlinear interaction between the two particles, and k is a positive constant.

The Legendre transform of (4.1) gives us the Hamiltonian function of the system described above, that is,

$$\begin{aligned} H(\mathbf{q}, \mathbf{p}) = {} & \frac{1}{2m}(p_1^2 + p_2^2 + p_3^2 + p_4^2) + \left(\frac{\gamma^2}{2m} - \frac{k}{2}\right)(q_1^2 + q_2^2 + q_3^2 + q_4^2) \\ & + \frac{\gamma}{m}(p_1 q_2 - p_2 q_1) + \frac{\gamma}{m}(p_3 q_4 - p_4 q_3) + f(\pi_1^i, \pi_2^i, \pi_3^i), \end{aligned} \tag{4.2}$$

This system has, for all values of the parameters γ and k, an equilibrium at the point $(q_1, q_3, q_2, q_4, p_1, p_3, p_2, p_4) = (\mathbf{0}, \mathbf{0})$. The linearization of the dynamics at that point is represented by the matrix (the coordinates are ordered as in the previous equality)

$$\mathcal{A}_k = \begin{pmatrix} -\frac{\gamma}{m}\mathbb{J}_4 & \frac{1}{m}\mathbb{I}_4 \\ \left(k - \frac{\gamma^2}{m}\right)\mathbb{I}_4 & -\frac{\gamma}{m}\mathbb{J}_4 \end{pmatrix}, \tag{4.3}$$

whose eigenvalues are

$$\lambda_k = \pm\frac{1}{m}\sqrt{km - 2\gamma^2 \pm 2\gamma\sqrt{\gamma^2 - km}}.$$

If we move the parameter k around the value $k_\circ = \gamma^2/m$ these eigenvalues undergo a Hamiltonian Hopf behavior like the one depicted in Fig. 1.1.

We now study the symmetries of the system. Note that after the assumptions on the interaction function f, the system is invariant under the canonical S^1 action given by the lifted action to the phase space of

$$(\varphi, \mathbf{q}) \longmapsto \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 & 0 \\ \sin\varphi & \cos\varphi & 0 & 0 \\ 0 & 0 & \cos\varphi & -\sin\varphi \\ 0 & 0 & \sin\varphi & \cos\varphi \end{pmatrix} \cdot \mathbf{q},$$

where $\mathbf{q} = (q_1, q_3, q_2, q_4)$, and by the transformation

$$\tau \cdot \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix} = \begin{pmatrix} q_1 \\ q_2 \\ -q_3 \\ -q_4 \end{pmatrix}.$$

The momentum map $\mathbf{K} : \mathbb{R}^8 \to \mathbb{R}$ associated with the S^1 action is given by the expression $\mathbf{K}(\mathbf{q}, \mathbf{p}) = p_3 q_1 - q_3 p_1 - p_2 q_4 + p_4 q_2$.

If we now look at the linearization (4.3) evaluated at the Hopf value $k_\circ = \gamma^2/m$ we see that in this case V_0 consists of the points of the form $(q_1, q_3, q_2, q_4, \mathbf{0})$. The S^1 action on V_0 generated by the semisimple part of $\mathcal{A}_{k_\circ}$ can be written as

$$(\theta, \mathbf{q}) \longmapsto \mathrm{e}^{-\theta \frac{\gamma}{m} \mathbb{J}_4} \cdot \mathbf{q}.$$

In order to better study the group actions on V_0 we will perform a linear change of variables. Let $(z_1, z_2, \bar{z_1}, \bar{z_2})$ be the new (complex) coordinates, given by

$$\begin{aligned} z_1 &= q_1 + q_4 + iq_2 - iq_3, \\ z_2 &= q_1 - q_4 + iq_2 + iq_3, \\ \bar{z_1} &= q_1 + q_4 - iq_2 + iq_3, \\ \bar{z_2} &= q_1 - q_4 - iq_2 - iq_3. \end{aligned} \tag{4.4}$$

If we take as new angles ψ_1 and ψ_2, defined by

$$\psi_1 = \varphi + \frac{\gamma}{m}\theta, \qquad \psi_2 = \varphi - \frac{\gamma}{m}\theta,$$

we realize that the previously introduced actions form a $O(2) \times S^1$ action on V_0 that takes the following convenient simple expression:

$$(\psi_1, \psi_2) \cdot (z_1, z_2) = (\mathrm{e}^{i\psi_1} z_1, \mathrm{e}^{i\psi_2} z_2) \qquad \text{and} \qquad \tau \cdot (z_1, z_2) = (z_2, z_1). \tag{4.5}$$

That is, we have shown that the system of two coupled harmonic oscillators subjected to a magnetic field, whose Hamiltonian is given by (4.2) can be taken as an example of Hamiltonian Hopf bifurcation with $O(2) \times S^1$ symmetry, which we will study in full generality in the following subsection.

4.2. RPOs in Hamiltonian Hopf bifurcation with $O(2)$ symmetry

Having the example in the previous section as a motivation we will study in what follows the RPOs that appear in a Hamiltonian Hopf bifurcation phenomenon in the presence of a $O(2)$ symmetry. The simplicity of this symmetry will allow us to explicitly write down the principal part of the reduced bifurcation equation in full generality, and to read off directly from it the RPOs that we are looking for.

We start by recalling that in the canonical coordinates introduced in (2.3) the principal part of the reduced bifurcation equation is, by Lemma 3.9, equal in our case to

$$\begin{aligned} B(v_0, \alpha, \lambda, \xi) = {} & (\lambda\sigma'(\lambda_\circ) + \alpha^2 \nu_\circ^2) v_0 - \xi^2 v_0 - 2\alpha\nu_\circ \mathbb{J}_4 \xi v_0 - 2\psi'(\lambda_\circ)\nu_\circ \alpha\lambda v_0 \\ & + 2\psi'(\lambda_\circ)\lambda \mathbb{J}_4 \xi v_0 + \mathbb{P}\mathbf{d}^4 h_{\lambda_\circ}(0)\left(v_0^{(4)}\right) + \text{h.o.t.} \end{aligned}$$

We now rewrite this expression in the coordinates $(z_1, z_2, \bar{z_1}, \bar{z_2})$ in which the $O(2) \times S^1$ action looks like (4.5). In doing so we need to express in these new coordinates the matrices ξ^2 and $\mathbb{J}_4\xi$, and this can be easily achieved by using the explicit expression of the change of variables (4.4). Indeed, a straightforward calculation shows that in those coordinates

$$\xi^2 = -\psi^2 \mathbb{I}_4 \quad \text{and} \quad \mathbb{J}_4\xi \equiv \psi \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

where $\psi \in \mathbb{R}$. Therefore, the first terms of the expansion of the reduced bifurcation equation are:

$$B(\mathbf{z}, \alpha, \lambda, \psi) = (\lambda\sigma'(\lambda_\circ) + \alpha^2\nu_\circ^2 + \psi^2) \begin{pmatrix} z_1 \\ z_2 \\ \bar{z_1} \\ \bar{z_2} \end{pmatrix} - 2\nu_\circ\alpha\psi \begin{pmatrix} z_1 \\ -z_2 \\ \bar{z_1} \\ -\bar{z_2} \end{pmatrix} + \begin{pmatrix} (a|z_1|^2 + b|z_2|^2)z_1 \\ (a|z_2|^2 + b|z_1|^2)z_2 \\ (a|z_1|^2 + b|z_2|^2)\bar{z_1} \\ (a|z_2|^2 + b|z_1|^2)\bar{z_2} \end{pmatrix} + \cdots, \tag{4.6}$$

where the coefficients a and b are related to the fourth order terms in the expansion of the Hamiltonian, that is, $\mathbb{P}\mathbf{d}^4 h_{\lambda_\circ}(0)\left(v_0^{(4)}\right)$. In order to keep the simplicity of the exposition we will assume that these two coefficients are non-zero and non-equal (otherwise we would have to go to higher orders in expression (4.6)). The RPOs that we are looking for are given by the solutions of the system of equations:

$$0 = (\lambda\sigma'(\lambda_\circ) + (\alpha\nu_\circ - \psi)^2 + a|z_1|^2 + b|z_2|^2 + \cdots)z_1, \tag{4.7}$$

$$0 = (\lambda\sigma'(\lambda_\circ) + (\alpha\nu_\circ + \psi)^2 + a|z_2|^2 + b|z_1|^2 + \cdots)z_2. \tag{4.8}$$

Since $\sigma'(\lambda_\circ) \neq 0$, (4.7) can be easily solved by dividing the expression by z_1 and then using the Implicit Function Theorem to define a function

$$\lambda \equiv \lambda(z_1, z_2, \alpha, \psi) = \frac{1}{\sigma'(\lambda_\circ)}\left[-(\alpha\nu_\circ - \psi)^2 - (a|z_1|^2 + b|z_2|^2) + \cdots\right] \tag{4.9}$$

that, when substituted into (4.7), solves it. Hence, plugging (4.9) into (4.8) we reduce the problem to that of solving a scalar equation, which can be done again via the Implicit Function Theorem: we divide (4.7) by z_1 and (4.8) by z_2 and subtract the resulting expressions to obtain:

$$0 = 4\nu_\circ\alpha\psi + (a - b)(|z_2|^2 - |z_1|^2) + \cdots \tag{4.10}$$

Since for equivariance reasons z_1 and z_2 always appear in the tail of the previous expression as combinations of $|z_1|^2$ and $|z_2|^2$, the hypotheses on the coefficients a and b allow us to solve this final scalar equation by defining a function

$$|z_2|^2 \equiv |z_2|^2\left(|z_1|^2, \alpha, \psi\right) = |z_1|^2 - \frac{4\nu_\circ\alpha\psi}{a - b} + \cdots \tag{4.11}$$

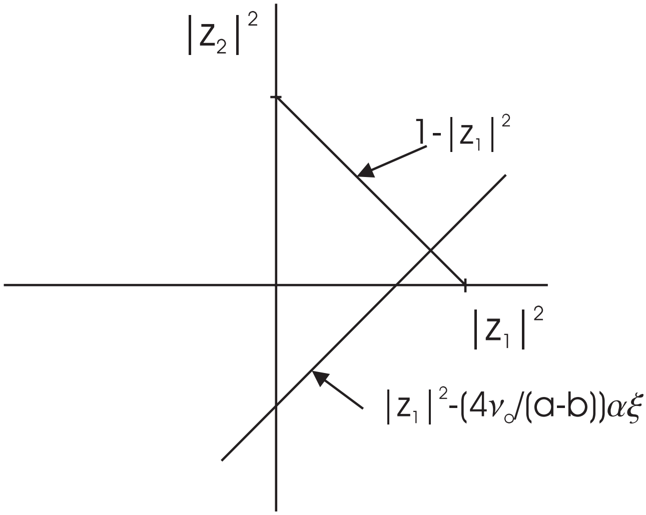

Fig. 4.1. Parametrization of the RPOs in the Hamiltonian Hopf bifurcation with $O(2)$-symmetry.

As we illustrate in Fig. 4.1, the solution (4.11) predicts, for each fixed value of the norm $|z_1|^2 + |z_2|^2$ a one-parameter family of RPOs that are obtained by varying the product $\alpha\psi$. More explicitly, and using Fig. 4.1, suppose that $|z_1|^2 + |z_2|^2 = 1$ and that the value $\alpha\psi$ is fixed, then, the intersection of the lines $|z_2|^2 = 1 - |z_1|^2$ and $|z_2|^2 = |z_1|^2 - \frac{4\nu_\circ\alpha\psi}{a-b}$ provides us with the abovementioned RPO.

Notice that all these RPOs cannot be predicted by merely using Theorem 3.4 since even though all the hypotheses needed in this result are fulfilled, it only predicts two RPOs for each value of the norm $|z_1|^2 + |z_2|^2$. Moreover, we cannot be sure that these are not just periodic motions, since the predicted orbits could lie in fixed spaces of maximal isotropy, thereby implying their periodicity.

Remark 4.1. In contrast with the Hamiltonian case, the Hopf bifurcation of non-trivial RPOs in the dissipative case with $O(2)$ symmetry is subjected to the presence of a cubic order degeneracy in the normal form. Unfolding this singularity leads to a codimension two bifurcation problem where the RPOs appear as a secondary branching from the primary branches of periodic orbits.

4.3. Hamiltonian Hopf bifurcation of RPOs for reduced integrable systems

The analysis performed in the previous section dealing with the Hamiltonian Hopf bifurcation with $O(2)$ symmetry is a particular case of a more general situation. Indeed, the main feature in that example was that it allowed us to carry out an explicit study of the reduced bifurcation equation was the coincidence of one half the dimension of the reduced space V_0 with the dimension of the symmetry group $O(2) \times S^1$. We will see in this section that whenever we are in the presence of this *reduced integrability hypothesis* an analysis in the same style can be performed.

More explicitly, all through this section we will be dealing with (V, ω, h_λ), a one-parameter family of G-Hamiltonian systems that satisfies conditions (H1)–(H4) such that if $4n$ is the dimension of the resonance space $U_{\nu_\circ}$ with primitive period $T_{\nu_\circ}$, then the rank of $G \times S^1$ equals n, that is, the maximal tori of the Lie group $G \times S^1$ have all dimension equal to n.

Let $\mathbb{T}^{n-1} \subset G$ be a maximal torus of G, and let $\xi \in \mathfrak{t}^{n-1}$ be an element in the Lie algebra of $\mathbb{T}^{n-1}$. As in the previous section, we can find coordinates in which the action of $\mathbb{T}^{n-1} \times S^1$ looks simple. Namely, there exists a set of complex coordinates $(z_1, \dots, z_n, \bar{z_1}, \dots, \bar{z_n})$ (and conjugates) for V_0 and a set of angular coordinates $(\xi_1, \dots, \xi_{n-1})$ for the torus T^{n-1}, for which the $\mathbb{T}^{n-1}$ action looks like

$$(\mathrm{e}^{i\xi_1}, \dots, \mathrm{e}^{i\xi_{n-1}}) \cdot (z_1, \dots, z_n) = (\mathrm{e}^{i\xi_1} z_1, \dots, \mathrm{e}^{i\xi_{n-1}} z_{n-1}, \mathrm{e}^{i(c_1\xi_1+\dots+c_{n-1}\xi_{n-1})} z_n),$$

where the coefficients $c_1, \dots, c_{n-1}$ are rational constants. If we incorporate the S^1 action using these complex coordinates, the $\mathbb{T}^{n-1} \times S^1$ action looks like

$$\begin{aligned}(\mathrm{e}^{ixi_1}, \dots, \mathrm{e}^{i\xi_{n-1}}, \mathrm{e}^{i\alpha}) \cdot (z_1, \dots, z_n)& \\ = (\mathrm{e}^{i(\xi_1+\alpha)} z_1, \dots, \mathrm{e}^{i(\xi_{n-1}+\alpha)} z_{n-1}, &\mathrm{e}^{i(c_1\xi_1+\cdots+c_{n-1}\xi_{n-1}+\alpha)} z_n).\end{aligned}$$

Let us now set

$$\psi_j = \xi_j + \alpha\ , \quad j = 1, \dots, n-1, \tag{4.12}$$

$$\psi_n = c_1\xi_1 + \cdots + c_{n-1}\xi_{n-1} + \alpha. \tag{4.13}$$

Under the condition

$$c_1 + \cdots + c_{n-1} \neq 1 \tag{4.14}$$

these relations define a change of coordinates on the n-dimensional torus $\mathbb{T}^{n-1} \times S^1$, and in these new coordinates the action can now be written in the very simple fashion

$$(\mathrm{e}^{i\psi_1}, \dots, \mathrm{e}^{i\psi_n}) \cdot (z_1, \dots, z_n) = (\mathrm{e}^{i\psi_1} z_1, \dots, \mathrm{e}^{i\psi_n} z_n). \tag{4.15}$$

Notice that under condition (4.14) the ring of invariant polynomials for this action on V_0 is generated by the quadratic invariants $\pi_j = z_j \bar{z}_j$, $j = 1, \dots, n$, and that the strata of this action are obtained by setting some of the z_j's equal to 0 while keeping the others different from 0. The orbit space for this action can be identified with the positive cone in $\mathbb{R}^n$ $\{(\pi_1, \dots, \pi_n) \ / \ \pi_j \geqq 0, \ j = 1, \dots, n\}$.

Recall now that in the canonical coordinates introduced in (2.3), the principal part of the reduced bifurcation equation is, by Lemma 3.9, equal to

$$\begin{aligned}B(v_0, \alpha, \lambda, \xi) = (\lambda\sigma'(\lambda_\circ) + \alpha^2\nu_\circ^2)v_0 - \xi^2 v_0 - 2\alpha\nu_\circ \mathbb{J}_{2n}\xi v_0 - 2\psi'(\lambda_\circ)\nu_\circ\alpha\lambda v_0& \\ + 2\psi'(\lambda_\circ)\lambda\mathbb{J}_{2n}\xi v_0 + C\left(v_0^{(3)}\right) + \text{h.o.t.},&\end{aligned} \tag{4.16}$$

From (4.15) it is clear that the matrices $\mathbb{J}_{2n}$, $\mathbb{J}_{2n}\xi$ and ξ^2 in (4.16) take, in these newly introduced coordinates, the form:

$$\mathbb{J}_{2n} = \begin{pmatrix} i & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & i \end{pmatrix}, \quad \mathbb{J}_{2n}\xi = \begin{pmatrix} -\psi_1 & \cdots & \cdots & 0 \\ \vdots & \ddots & & \vdots \\ \vdots & & -\psi_{n-1} & \vdots \\ 0 & \cdots & \cdots & -\psi_n \end{pmatrix},$$

$$\xi^2 = -\begin{pmatrix} \psi_1^2 & \cdots & \cdots & 0 \\ \vdots & \ddots & & \vdots \\ \vdots & & \psi_{n-1}^2 & \vdots \\ 0 & \cdots & \cdots & \psi_n^2 \end{pmatrix}.$$

Using these new coordinates and the $\mathbb{T}^{n-1} \times S^1$ equivariance of B, we rewrite the components of (4.16) (we omit the complex conjugate part). For $i \in \{1, \ldots, n\}$ we have:

$$B_i(z, \alpha, \lambda, \xi) = \left(\lambda\sigma'(\lambda_\circ) + \psi_i^2 + \hat{C}_i\left(|z_1|^2, \ldots, |z_n|^2\right) + \text{h.o.t.}\right) z_i,$$

where

$$\hat{C}_i\left(|z_1|^2, \ldots, |z_n|^2\right) = \hat{c}_{i1}|z_1|^2 + \cdots + \hat{c}_{i1}|z_n|^2, \qquad \hat{c}_{i1} \in \mathbb{R}, \quad i, j \in \{1, \ldots, n\}.$$

We can now state a theorem about the bifurcation of RPOs.

Theorem 4.2. *Let* (V, ω, h_λ) *be a one-parameter family of* G*-Hamiltonian systems that satisfies conditions* (H1)–(H4)*. Suppose that:* (i) *the dimension of* V_0 *equals twice the rank* n *of* $G \times S^1$*;* (ii) *the condition* (4.14) *on the torus action is satisfied. Then, if the matrix*

$$\Delta = \left(\hat{c}_{nj} - \hat{c}_{ij}\right), \quad 1 \leqq i \leqq n, \ 1 \leqq j \leqq n-1,$$

has maximal rank $n-1$*, there exists a family of RPOs with* n *different frequencies which bifurcates from the trivial solution as* λ *crosses* λ_0*.*

Remark 4.3. The condition on the matrix Δ is not generic because the values of the coefficients $\hat{c}_{ij}$ are constrained by the G-equivariance of the operator C. More specifically if the G-action was purely toral the coefficients $\hat{c}_{ij}$ would be independent and Δ would generically have maximal rank. However, if G contains other elements, they can force a nontrivial relation among these coefficients that may result in a loss of the generic maximality in the rank of Δ. In the framework of the non-Hamiltonian Hopf bifurcation with $SO(3)$ symmetry, a generic relative periodic solution has been found [Le97] for which neither the integrability hypothesis nor the maximality feature in rank(Δ) hold. Nevertheless, in Section 4.4 we show that in the context of Hamiltonian systems with $SO(3)$ symmetry nontrivial families of RPOs can be found by applying Theorem 4.2.

Remark 4.4. In the non-symmetric case (that is $G = \{e\}$) the condition on the maximality of $\operatorname{rank}(\Delta)$ amounts, in the notation of [vdM90], to having a *nondegenerate* Hamiltonian Hopf bifurcation. The degenerate non-symmetric Hamiltonian Hopf bifurcation has been studied in [vdM90]. The symmetric counterpart of this work will be the subject of a future study.

Proof. Since we are looking for solutions with $z_i \neq 0$, for all i, we can factor out z_i in each equation $B_i = 0$. The resulting equations read, for $i \in \{1, \ldots, n\}$,

$$0 = \lambda\sigma'(\lambda_\circ) + \psi_i^2 + \hat{C}_i\left(|z_1|^2, \ldots, |z_n|^2\right) + \text{h.o.t.} \tag{4.17}$$

These equations are simply those that we would have obtained by projecting first (4.16) on the orbit space corresponding to the toral action; in all that follows we will denote $|z_i|^2$ by π_i. Since by hypothesis (H4), $\sigma'(\lambda_\circ) \neq 0$, we can solve any one of these equations for λ. Let us do so for the equation with $i = n$. By replacing λ by the resulting expression in the remaining equations, we have reduced the problem to solving a system of $n - 1$ equations which, at leading order, have the form

$$0 = \psi_i^2 - \psi_n^2 + (\hat{c}_{i1} - \hat{c}_{n1})\pi_1 + \cdots + (\hat{c}_{in} - \hat{c}_{nn})\pi_n + \text{h.o.t.} \tag{4.18}$$

If the matrix Δ has maximal rank, we obtain a unique family of solutions of the system (4.18), for which $n - 1$ of the π_i's depend smoothly on the remaining one and on the parameters ψ_j, $j = 1, \ldots, n$. In order to fix notation, let us assume without loss of generality that we have obtained $\pi_i = \pi_i(\psi_1, \ldots, \psi_n, \pi_n)$ for $i = 1, \ldots, n - 1$. These solutions still have to lie inside the orbit space, that is, we still have to check the additional conditions $\pi_i \geqq 0$. However, since the ψ_j's are free parameters, the quantities $\psi_i^2 - \psi_n^2$ can take any real value. Therefore, if we set $z_n = 0$, we can always find values for the ψ_j's such that $\pi_i > 0$ for $i = 1, \ldots, n - 1$. These inequalities are still satisfied if the ψ_j's are close enough to these values and $\pi_n > 0$ is close enough to 0. Finally, since these solutions lie on the principal stratum for the action of $\mathbb{T}^{n-1} \times S^1$, the corresponding RPOs have n different frequencies which depend smoothly on λ. □

Remark 4.5. In this proof, we could have directly solved the system of equations (4.17) for the variables ψ_j^2 by the Implicit Function Theorem, hence proving the existence of a family of bifurcated solutions. However, there is no guarantee that any of these solutions will not be just periodic orbits.

Remark 4.6. In the problem with $O(2)$ symmetry analyzed in Section 4.2, we have $n = 2$, $c_1 = -1$, $\hat{c}_{11} = \hat{c}_{22}$ and $\hat{c}_{12} = \hat{c}_{21}$. The hypotheses of Theorem 4.2 are therefore generically satisfied in this case.

Remark 4.7. As it was already the case with Theorem 3.4, Theorem 4.2 still applies if instead of $G \times S^1$ acting in V_0, we consider the group $N(H)/H$ acting in V_0^H, where H is an isotropy subgroup of the $G \times S^1$ action. In the next section we shall see an application of this remark.

4.4. *Hamiltonian Hopf bifurcation with $SO(3)$ symmetry*

Hopf bifurcation problems with $SO(3)$ symmetry for dissipative systems have been investigated by several authors in the case in which the eigenspaces associated with the critical eigenvalues is the direct sum of twice the five-dimensional (real) irreducible representation of $SO(3)$ (see [GoSt85]), [IoRo89], [MRS88], and [Le97]). This is the simplest possible case with $SO(3)$ symmetry which does not reduce to Hopf bifurcation with either trivial or $O(2)$ symmetry. Nevertheless, it leads to a normal form in a ten-dimensional real vector space. The list of solutions with maximal isotropy, hence purely periodic ones, has been given in [GoSt85] and in [MRS88]. However, the most interesting feature of this problem is the possibility of having a bifurcated branch of RPOs in a six-dimensional subspace. This was first found by [IoRo89]. Another approach was taken by [Le97] (using orbit space reduction) who did not recover the result of [IoRo89]. This remark shows the level of difficulty found in obtaining direct branching of RPOs via Hopf bifurcation for equivariant vector fields. In the Hamiltonian context, a related work by Haaf, Roberts & Stewart [HRS92] has shown the existence of families of periodic orbits with maximal isotropy for a Hamiltonian in $\mathbb{R}^{10}$ which is invariant under the same $SO(3)$ action.

In what follows we investigate the Hamiltonian Hopf bifurcation with $SO(3)$ symmetry, when the subspaces V_0 and V_1 are associated with this ten-dimensional representation and we shall see that, in this case, Theorem 4.2 applies and shows the existence of several families of RPOs.

Let

$$V = V_0 \oplus V_1 \simeq \mathbb{R}^{10}.$$

We identify $\mathbb{R}^{10}$ with $\mathbb{R}^5 \otimes \mathbb{C}$ and consider the action of $SO(3)$ on $\mathbb{R}^5$ given by its irreducible representation on the space of spherical harmonics of degree 2. Equivalently, we may identify $\mathbb{R}^5$ with the space W of 3×3 real symmetric matrices with trace 0, and consider the action of $SO(3)$ on W defined by

$$\rho_A(M) = A^{-1} M A, \quad A \in SO(3), \ M \in W.$$

This definition extends naturally to $\mathbb{R}^5 \otimes \mathbb{C}$ with the same formula, M now having complex coefficients. We shall therefore identify in all that follows V_0 with $W \otimes \mathbb{C}$. The S^1-action on V_0 is simply defined as multiplication by $e^{i\theta}$ in $\mathbb{C}$, that is, $\theta \cdot M := e^{i\theta} M$.

Any $M \in W \otimes \mathbb{C}$ decomposes uniquely as

$$M = \sum_{m=-2}^{2} z_m B_m,$$

where

$$B_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & i \\ 1 & i & 0 \end{pmatrix}, \quad B_{-1} = \overline{B}_1, \tag{4.19}$$

$$B_2 = \begin{pmatrix} 1 & i & 0 \\ i & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B_{-2} = \overline{B}_2. \tag{4.20}$$

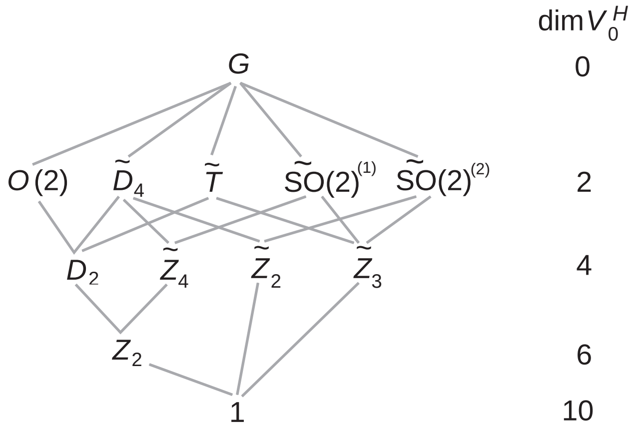

Fig. 4.2. Isotropy lattice of the $SO(3) \times S^1$ action.

We now list the isotropy types for the action of $G = SO(3) \times S^1$ on V_0 that we have just defined. We use the presentation and results of [HRS92]. Figure 4.2 shows the isotropy lattice of the G-action and the dimension of the corresponding fixed-point subspaces.

Notation. The group $\tilde{H}$ is a subgroup isomorphic to $H \subset O(3)$ but such that $\tilde{H} \cap S^1 \neq 1$ (here 1 is the trivial group). In particular, $\tilde{\mathbb{Z}}_n$ is the group generated by $(R_n, -2\pi/n) \in SO(3) \times S^1$, where R_n is a rotation of angle $2\pi/n$.

By Corollary 3.10, for any subgroup H with $\dim(V_0^H) = 2$ there exists a branch of periodic solutions having this symmetry. Let us now consider those isotropy subgroups having a fixed-point subspace of dimension 4. The largest subgroup acting faithfully in V_0^H is $N(H)/H$. We list below the different $N(H)/H$ for the non-maximal isotropy subgroups:

1. $N(D_2)/D_2 \simeq D_3 \times S^1$;
2. $N(\tilde{\mathbb{Z}}_4)/\tilde{\mathbb{Z}}_4 \simeq N(\tilde{\mathbb{Z}}_2)/\tilde{\mathbb{Z}}_2 \simeq O(2) \times S^1$;
3. $N(\tilde{\mathbb{Z}}_3)/\tilde{\mathbb{Z}}_3 \simeq SO(2) \times S^1$;
4. $N(\mathbb{Z}_2)/\mathbb{Z}_2 \simeq O(2) \times S^1$;
5. $N(1) = G$.

In case 1 we see that solutions with isotropy D_2 are always periodic. Case 2 corresponds to the problem described in Section 4.2 (Hopf bifurcation with $O(2)$ symmetry). It was noticed in [HRS92] that the equations in V_0^H do not degenerate despite the fact that they come from a system with higher symmetry, which leads us to conclude that families of RPOs with two frequencies and with spatio-temporal symmetry $\tilde{\mathbb{Z}}_4$ as well as $\tilde{\mathbb{Z}}_2$ do generically bifurcate. Case 3 falls in the framework of Section 4.3, since the symmetry group is $SO(2) \times S^1$. However, because the equations in $V_0^{\tilde{\mathbb{Z}}_3}$ are the restriction in that subspace of a system with higher symmetry in V_0, we need to compute the cubic order terms in order to insure that no "hidden" degeneracy occurs. We use the argument proved in [HRS92]; we can choose $\tilde{\mathbb{Z}}_3$ so that, by introducing complex coordinates,

$$V_0^{\tilde{\mathbb{Z}}_3} = \{z_1 B_1 + z_2 B_{-2},\ (z_1, z_2) \in \mathbb{C}^2\} \tag{4.21}$$

and the action of $SO(2)$ is then defined by

$$\phi \cdot (w, z) = (e^{i\phi}, e^{-2i\phi}).$$

With the notation of Section 4.3, we therefore have

$$\psi_1 = \phi \text{ and } c_1 = -2.$$

The expression for the cubic G-equivariant terms is

$$C(M^{(3)}) = b_1 \mathrm{tr}(M\bar{M})M + b_2 \mathrm{tr}(M^2)\bar{M} + b_3 \left(M^2\bar{M} + \bar{M}M^2 - \tfrac{2}{3}\mathrm{tr}(M^2\bar{M})Id\right);$$

with b_j real coefficients depending on the specific Hamiltonian at hand. Setting $M = z_1 B_1 + z_2 B_{-2}$ we obtain, after calculation in $V_0^{\tilde{\mathbb{Z}}_3}$, the expression

$$\begin{aligned} C(M^{(3)}) = {} & 4\left((b_1 + \tfrac{1}{2}b_3)|z_1|^2 + (b_1 + b_3)|z_2|^2\right) z_1 B_1 \\ & + 4\left((b_1 + b_3)|z_1|^2 + b_1|z_2|^2\right) z_2 B_{-2}. \end{aligned}$$

Let us now check whether the 1×2 matrix Δ of Theorem 4.2 has maximal rank. From the above expression we deduce that

$$\hat{c}_{11} - \hat{c}_{21} = -2b_3, \quad \hat{c}_{12} - \hat{c}_{22} = 4b_3.$$

Therefore the maximality hypothesis is satisfied if and only if $b_3 \neq 0$ (which is a generic condition).
Cases 4 and 5 are beyond the range of applicability of Theorem 4.2.

5. Appendix

5.1. On the invariance properties of the resonance subspace

In what follows we will sketch the proof of some of the facts about the invariance properties of the resonance subspace mentioned in the preliminaries section when (V, ω) is a symplectic representation space of the Lie group G and the Hamiltonian vector field A is G-equivariant.

The resonance subspace $U_{\nu_\circ}$ is G-invariant. Let $A = A_s + A_n$ be the Jordan-Chevalley decomposition of A. Since by hypothesis A is G-equivariant, if $\Phi : G \times V \to V$ denotes the G action, for any $g \in G$, we know that $\Phi_g A = A\Phi_g$. Equivalently, $\Phi_g A_s + \Phi_g A_n = A_s\Phi_g + A_n\Phi_g$, and hence $\Phi_g A_n \Phi_{g^{-1}} + \Phi_g A_s \Phi_{g^{-1}} = A_n + A_s$. Since $\Phi_g A_n \Phi_{g^{-1}}$ is nilpotent, $\Phi_g A_s \Phi_{g^{-1}}$ issemisimple,

$[\Phi_g A_n \Phi_{g^{-1}}, \Phi_g A_s \Phi_{g^{-1}}] = 0$, and the Jordan-Chevalley decomposition is unique, we have

$$\Phi_g A_n \Phi_{g^{-1}} = A_n \qquad \text{and} \qquad \Phi_g A_s \Phi_{g^{-1}} = A_s,$$

necessarily. This implies the G-invariance of $U_{\nu_\circ} = \ker(e^{A_s T_{\nu_\circ}} - I)$. Indeed, let $v \in U_{\nu_\circ}$. Hence, $e^{A_s T_{\nu_\circ}} v = v$. At the same time, for any $g \in G$,

$$e^{A_s T_{\nu_\circ}}(\Phi_g v) = \Phi_g e^{A_s T_{\nu_\circ}} v = \Phi_g v,$$

hence $\Phi_g v \in U_{\nu_\circ}$, that is, $U_{\nu_\circ}$ is G-invariant.

The S^1 action and the G action on $U_{\nu_\circ}$ commute. Let $\Psi : S^1 \times U_{\nu_\circ} \to U_{\nu_\circ}$ be the S^1-action on $U_{\nu_\circ}$. For any $g \in G$ and any $\theta \in S^1$:

$$\Phi_g \Psi_\theta = \Phi_g e^{\theta A_s} = e^{\theta A_s} \Phi_g = \Psi_\theta \Phi_g,$$

as required.

5.2. Proof of Lemma 3.8

The defining relation (3.4) of the function v_1 implies that for any $v_0 \in V_0$, $\alpha, \lambda \in \mathbb{R}$, and $\xi \in \mathfrak{h}$ we have

$$(\mathbb{I} - \mathbb{P}) \nabla_{U_{\nu_\circ}} (\hat{h}_\lambda - \mathbf{J}^{1+\alpha} - \mathbf{K}^\xi)(v_0 + v_1(v_0, \alpha, \lambda, \xi)) = 0.$$

Consequently, for any $w_1 \in V_1$:

$$\begin{aligned} 0 &= \left.\frac{d}{dt}\right|_{t=0} \langle \nabla_{U_{\nu_\circ}} (\hat{h}_\lambda - \mathbf{J}^{1+\alpha} - \mathbf{K}^\xi)(t v_0 + v_1(t v_0, \alpha, \lambda, \xi)), w_1 \rangle \\ &= \left.\frac{d}{dt}\right|_{t=0} \mathbf{d}(\hat{h}_\lambda - \mathbf{J}^{1+\alpha} - \mathbf{K}^\xi)(t v_0 + v_1(t v_0, \alpha, \lambda, \xi)) \cdot w_1 \\ &= \mathbf{d}^2 (\hat{h}_\lambda - \mathbf{J}^{1+\alpha} - \mathbf{K}^\xi)(0)(v_0 + D_{V_0} v_1(0, \alpha, \lambda, \xi) \cdot v_0, w_1). \end{aligned}$$

If we use (2.6) and (2.14), the previous expression can be expressed in matrix form as

$$\begin{aligned} 0 &= (0, w_1) \\ &\cdot \begin{pmatrix} \sigma(\lambda)\mathbb{I}_{2n} & \tau(\lambda)\mathbb{I}_{2n} + (\psi(\lambda) - (1+\alpha))\mathbb{J}_{2n} - \xi \\ \tau(\lambda)\mathbb{I}_{2n} - (\psi(\lambda) - (1+\alpha))\mathbb{J}_{2n} + \xi & \rho(\lambda)\mathbb{I}_{2n} \end{pmatrix} \\ &\cdot \begin{pmatrix} v_0 \\ D_{V_0} v_1(0, \alpha, \lambda, \xi) \cdot v_0 \end{pmatrix} \\ &= w_1^T \left[\tau(\lambda)\mathbb{I}_{2n} - (\psi(\lambda) - (1+\alpha))\mathbb{J}_{2n} + \xi + \rho(\lambda) D_{V_0} v_1(0, \alpha, \lambda, \xi) \right] v_0. \end{aligned}$$

Given that the previous equation is valid for no matter what $v_0 \in V_0$ and $w_1 \in V_1$, we can conclude that

$$D_{V_0} v_1(0, \alpha, \lambda, \xi) = \frac{\psi(\lambda) - (1+\alpha)}{\rho(\lambda)} \mathbb{J}_{2n} - \frac{\tau(\lambda)}{\rho(\lambda)} \mathbb{I}_{2n} - \frac{\xi}{\rho(\lambda)},$$

as required. □

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