

The symplectic reduced spaces of a Poisson action

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Abstract

During the last thirty years, symplectic or Marsden–Weinstein reduction has been a major tool in the construction of new symplectic manifolds and in the study of mechanical systems with symmetry. This procedure has been traditionally associated to the canonical action of a Lie group on a symplectic manifold, in the presence of a momentum map. In this Note we show that the symplectic reduction phenomenon has much deeper roots. More specifically, we will find symplectically reduced spaces purely within the Poisson category under hypotheses that do not necessarily imply the existence of a momentum map. In other words, the right category to obtain symplectically reduced spaces is that of Poisson manifolds acted upon canonically by a Lie group. *To cite this article: J.-P. Ortega, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 999–1004.* © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Les espaces réduits symplectiques d'une action de Poisson

Résumé

Pendant les trente dernières années, la réduction symplectique (aussi appelée de Marsden–Weinstein) a joué un rôle majeur lors de la construction de nouvelles variétés symplectiques et dans l'étude des systèmes mécaniques symétriques. Ce procédé a été traditionnellement associé à l'action canonique d'un groupe de Lie sur une variété symplectique, en présence d'une application moment. Dans cette Note, nous montrerons que le phénomène de la réduction symplectique a des racines beaucoup plus profondes. Plus spécifiquement, nous trouverons des espaces réduits symplectiques à l'intérieur de la catégorie des variétés de Poisson sous des hypothèses qui n'impliquent pas forcément l'existence d'une application moment. Autrement dit, la catégorie la plus générale pour l'obtention des espaces réduits symplectiques est celle des variétés de Poisson munies de l'action canonique d'un groupe de Lie. *Pour citer cet article : J.-P. Ortega, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 999–1004.* © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Version française abrégée

Soit $(M, \{\cdot, \cdot\})$ une variété de Poisson et G un groupe de Lie qui agit sur M d'une façon canonique et propre. Soit A'_G la distribution lisse et intégrable sur M définie par l'identité $A'_G := \{X_f \mid f \in C^\infty(M)^G\}$, où $C^\infty(M)^G$ est l'ensemble des fonctions lisses G -invariantes sur M et X_f est le champ de vecteurs hamiltonien associé à la fonction f . L'application moment optimale $\mathcal{J} : M \rightarrow M/A'_G$ a été définie en [12] comme la projection canonique de M sur l'espace des feuilles M/A'_G de A'_G .

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Par construction, les surfaces de niveau de \mathcal{J} sont des sous-variétés immergées préservées par la dynamique hamiltonienne G -équivariante sur la variété de Poisson $(M, \{\cdot, \cdot\})$. En plus, on peut définir une action du groupe G sur l'espace de feuilles M/A'_G par rapport à laquelle \mathcal{J} est équivariante.

Le résultat principal de cette Note est la démonstration du théorème suivant.

THÉORÈME 0.1. – *Soit $(M, \{\cdot, \cdot\})$ une variété de Poisson et G un groupe de Lie qui agit sur M d'une façon canonique et propre. Soit $\mathcal{J} : M \rightarrow M/A'_G$ l'application moment optimale associée à cette action. Alors, pour un élément $\rho \in M/A'_G$ quelconque dont le sous-groupe d'isotropie G_ρ agit proprement sur $\mathcal{J}^{-1}(\rho)$, l'espace des orbites $\mathcal{J}^{-1}(\rho)/G_\rho$ est une variété quotient régulière (c.-à-d., la projection canonique $\pi_\rho : \mathcal{J}^{-1}(\rho) \rightarrow \mathcal{J}^{-1}(\rho)/G_\rho$ est une submersion surjective). En plus, c'est une variété symplectique, avec la forme symplectique naturelle ω_ρ définie par l'identité*

$$\pi_\rho^* \omega_\rho(m)(X_f(m), X_h(m)) = \{f, h\}(m), \quad \text{pour } m \in \mathcal{J}^{-1}(\rho) \text{ et } f, h \in C^\infty(M)^G.$$

1. Introduction

Let (M, ω) be a symplectic manifold and G be a Lie group that acts freely and properly on M . We will assume that this action is canonical that is, it preserves the symplectic form and that it has an equivariant momentum map $\mathbf{J} : M \rightarrow \mathfrak{g}^*$ associated. Marsden and Weinstein [8] showed that for any value $\mu \in \mathbf{J}(M)$ with coadjoint isotropy subgroup G_μ , the quotient $\mathbf{J}^{-1}(\mu)/G_\mu$ is a smooth symplectic manifold with a symplectic structure naturally inherited from that in M . This procedure can be reproduced when, instead of a \mathfrak{g}^* -valued momentum map, we have a G -valued momentum map in the sense of Alekseev et al. [9,1].

The study of symplectic reduction in the absence of the freeness hypothesis on the G -action has given rise to the so called *Singular Reduction Theory* which has been spelled out over the years in a series of works. See [2,14,3,10,5,13], and references therein.

The first effort to perform symplectic reduction without momentum maps was carried out in [12] by using the so called *optimal momentum map*. Nevertheless, in the requirements of the reduction theorem formulated in that paper there is a “closedness hypothesis” that is reminiscent at some level of the existence of a standard (\mathfrak{g}^* or G -valued) momentum map.

In this Note we will formulate a symplectic reduction theorem that does not require this hypothesis and that at the same time works in the Poisson category. More specifically, we will show that the Marsden–Weinstein quotients constructed using the (always available) optimal momentum map associated to a canonical Lie group action on the Poisson manifold $(M, \{\cdot, \cdot\})$ are smooth symplectic manifolds, provided that the group action satisfies a customary properness hypothesis.

2. The optimal momentum map and the momentum space

The optimal momentum map was introduced in [12] as a general method to find the conservation laws associated to the symmetries of a Poisson system encoded in the canonical action of a Lie group. We recall its definition. Let $(M, \{\cdot, \cdot\})$ be a Poisson manifold and G be a Lie group that acts properly on M by Poisson diffeomorphisms via the left action $\Phi : G \times M \rightarrow M$. The group of canonical transformations associated to this action will be denoted by $A_G := \{\Phi_g : M \rightarrow M \mid g \in G\}$ and the canonical projection of M onto the orbit space by $\pi_{A_G} : M \rightarrow M/A_G = M/G$. Let A'_G be the distribution on M defined by the relation $A'_G(m) := \{X_f(m) \mid f \in C^\infty(M)^G\}$, for all $m \in M$. The symbol X_f denotes the Hamiltonian vector field associated to the function $f \in C^\infty(M)$. Depending on the context, the distribution A'_G is called the *G -characteristic distribution* or the *polar distribution* defined by A_G [11]. A'_G is a smooth integrable generalized distribution in the sense of Stefan and Sussman [15–17]. The *optimal momentum map* \mathcal{J} is defined as the canonical projection onto the leaf space of A'_G , that is, $\mathcal{J} : M \rightarrow M/A'_G$. By its very definition, the levels sets of \mathcal{J} are preserved by the Hamiltonian flows associated to G -invariant Hamiltonian functions and \mathcal{J} is *universal* with respect to this property, that is, any other map whose level

sets are preserved by G -equivariant Hamiltonian dynamics factors necessarily through \mathcal{J} . By construction, the fibers of \mathcal{J} are the leaves of an integrable generalized distribution and thereby *initial immersed submanifolds* of M [6]. Recall that we say that N is an initial submanifold of M when the injection $i : N \rightarrow M$ is a smooth immersion that satisfies that for any manifold Z , a mapping $f : Z \rightarrow N$ is smooth iff $i \circ f : Z \rightarrow M$ is smooth. We summarize this and other elementary properties of the fibers of \mathcal{J} in the following proposition.

PROPOSITION 2.1. – *Let $(M, \{\cdot, \cdot\})$ be a Poisson manifold and G be a Lie group that acts properly and canonically on M . Let $\mathcal{J} : M \rightarrow M/A'_G$ be the associated optimal momentum map. Then for any $\rho \in M/A'_G$ we have that:*

- (i) *The level set $\mathcal{J}^{-1}(\rho)$ is an immersed initial submanifold of M .*
- (ii) *There is a unique symplectic leaf \mathcal{L} of $(M, \{\cdot, \cdot\})$ such that $\mathcal{J}^{-1}(\rho) \subset \mathcal{L}$.*
- (iii) *Let $m \in M$ be an arbitrary element of $\mathcal{J}^{-1}(\rho)$. Then, $\mathcal{J}^{-1}(\rho) \subset M_{G_m}$, with $M_{G_m} := \{z \in M \mid G_z = G_m\}$.*

In the sequel we will denote by \mathcal{L}_ρ the unique symplectic leaf of M that contains $\mathcal{J}^{-1}(\rho)$. Notice that as \mathcal{L}_ρ is also an immersed initial submanifold of M , the injection $i_{\mathcal{L}_\rho} : \mathcal{J}^{-1}(\rho) \hookrightarrow \mathcal{L}_\rho$ is smooth.

The leaf space M/A'_G is called the *momentum space* of \mathcal{J} . We will consider it as a topological space with the quotient topology. Let $m \in M$ be arbitrary such that $\mathcal{J}(m) = \rho \in M/A'_G$. Then, for any $g \in G$, the map $\Psi_g(\rho) = \mathcal{J}(g \cdot m) \in M/A'_G$ defines a continuous G -action on M/A'_G with respect to which \mathcal{J} is G -equivariant. Notice that since this action is not smooth and M/A'_G is not Hausdorff in general, there is no guarantee that the isotropy subgroups G_ρ are closed, and therefore embedded, subgroups of G . However, there is still something that we can say:

PROPOSITION 2.2. – *Let G_ρ be the isotropy subgroup of the element $\rho \in M/A'_G$ associated to the G -action on M/A'_G that we just defined. Then:*

- (i) *There is a unique smooth structure on G_ρ for which this subgroup becomes an initial Lie subgroup of G with Lie algebra \mathfrak{g}_ρ given by $\mathfrak{g}_\rho = \{\xi \in \mathfrak{g} \mid \xi_M(m) \in T_m\mathcal{J}^{-1}(\rho), \text{ for all } m \in \mathcal{J}^{-1}(\rho)\}$.*
- (ii) *With this smooth structure for G_ρ , the left action $\Phi^\rho : G_\rho \times \mathcal{J}^{-1}(\rho) \rightarrow \mathcal{J}^{-1}(\rho)$ defined by $\Phi^\rho(g, z) := \Phi(g, z)$ is smooth.*
- (iii) *This action has fixed isotropies, that is, if $z \in \mathcal{J}^{-1}(\rho)$ then $(G_\rho)_z = G_z$, and $G_m = G_z$ for all $m \in \mathcal{J}^{-1}(\rho)$.*

Proof. – (i) It is a straightforward corollary of Definition 3 and Proposition 9 in page 290 of [4]. Indeed, we can use that result to conclude the existence of a unique smooth structure for G_ρ with which it becomes an initial subgroup of G with Lie algebra $\mathfrak{g}_\rho = \{\xi \in \mathfrak{g} \mid \text{there exists a smooth curve } c : \mathbb{R} \rightarrow G_\rho \text{ such that } c(0) = e \text{ and } c'(0) = \xi\}$. An elementary argument shows that $\mathfrak{g}_\rho = \{\xi \in \mathfrak{g} \mid \exp t\xi \cdot m \in \mathcal{J}^{-1}(\rho) \text{ for all } m \in \mathcal{J}^{-1}(\rho), t \in \mathbb{R}\} = \{\xi \in \mathfrak{g} \mid \xi_M(m) \in T_m\mathcal{J}^{-1}(\rho), \text{ for all } m \in \mathcal{J}^{-1}(\rho)\}$.

(ii) As $\mathcal{J}^{-1}(\rho)$ is an initial submanifold of M and $i_\rho \circ \Phi^\rho$ is smooth, with $i_\rho : \mathcal{J}^{-1}(\rho) \hookrightarrow M$ the natural inclusion, then Φ^ρ is also smooth. (iii) is a straightforward consequence of the definitions. \square

3. The reduction theorem

We will now introduce our main result. In the statement we will denote by $\pi_\rho : \mathcal{J}^{-1}(\rho) \rightarrow \mathcal{J}^{-1}(\rho)/G_\rho$ the canonical projection onto the orbit space of the G_ρ -action on $\mathcal{J}^{-1}(\rho)$ defined in Proposition 2.2.

THEOREM 3.1 (Symplectic reduction by Poisson actions). – *Let $(M, \{\cdot, \cdot\})$ be a smooth Poisson manifold and G be a Lie group acting canonically and properly on M . Let $\mathcal{J} : M \rightarrow M/A'_G$ be the optimal momentum map associated to this action. Then, for any $\rho \in M/A'_G$ whose isotropy subgroup G_ρ acts properly on $\mathcal{J}^{-1}(\rho)$, the orbit space $M_\rho := \mathcal{J}^{-1}(\rho)/G_\rho$ is a smooth symplectic regular quotient manifold with symplectic form ω_ρ defined by:*

$$\pi_\rho^* \omega_\rho(m)(X_f(m), X_h(m)) = \{f, h\}(m), \quad \text{for any } m \in \mathcal{J}^{-1}(\rho) \text{ and any } f, h \in C^\infty(M)^G. \quad (1)$$

Remark 1. – Let $i_{\mathcal{L}_\rho} : \mathcal{J}^{-1}(\rho) \hookrightarrow \mathcal{L}_\rho$ be the natural smooth injection of $\mathcal{J}^{-1}(\rho)$ into the symplectic leaf $(\mathcal{L}_\rho, \omega_{\mathcal{L}_\rho})$ of $(M, \{\cdot, \cdot\})$ in which it is sitting. As \mathcal{L}_ρ is an initial submanifold of M , the injection $i_{\mathcal{L}_\rho}$ is a smooth map. The form ω_ρ can also be written in terms of the symplectic structure of the leaf \mathcal{L}_ρ as $\pi_\rho^* \omega_\rho = i_{\mathcal{L}_\rho}^* \omega_{\mathcal{L}_\rho}$. In view of this remark we can obtain the standard Symplectic Stratification Theorem of Poisson manifolds as a straightforward corollary of Theorem 3.1 by taking the group $G = \{e\}$. In that case the distribution A'_G coincides with the characteristic distribution of the Poisson manifold and the level sets of the optimal momentum map, and thereby the symplectic quotients M_ρ , are exactly the symplectic leaves. We explicitly point this out in our next statement.

COROLLARY 3.2 (Symplectic Stratification Theorem). – *Let $(M, \{\cdot, \cdot\})$ be a smooth Poisson manifold. Then, M is the disjoint union of the maximal integral leaves of the integrable distribution D given by $D(m) := \{X_f(m) \mid f \in C^\infty(M)\}$, $m \in M$. These leaves are symplectic initial submanifolds of M .*

Remark 2. – The only extra hypothesis in the statement of Theorem 3.1 with respect to the hypotheses used in the classical reduction theorems is the properness of the G_ρ -action on $\mathcal{J}^{-1}(\rho)$. The next example will show that this is a real hypothesis in the sense that the properness of the G_ρ -action is not automatically inherited from the properness of the G -action on M , as it used to be the case in the presence of a standard momentum map (see [12]). From this reduction point of view we can think of the presence of a standard momentum map as an extra integrability feature of the G -characteristic distribution that makes its integrable leaves imbedded (and not just initial) submanifolds of M and their isotropy subgroups automatically closed.

Example 1 (On the properness of the G_ρ -action). – As we announced in the previous remark, we now present a situation where the G_ρ -action on $\mathcal{J}^{-1}(\rho)$ is not proper while the G -action on M satisfies this condition. Let $M := \mathbb{T}^2 \times \mathbb{T}^2$ be the product of two two-tori whose elements we will denote by the four-tuples $(e^{i\theta_1}, e^{i\theta_2}, e^{i\psi_1}, e^{i\psi_2})$. We endow M with the symplectic structure ω defined by $\omega := \mathbf{d}\theta_1 \wedge \mathbf{d}\theta_2 + \sqrt{2} \mathbf{d}\psi_1 \wedge \mathbf{d}\psi_2$. We now consider the canonical two-torus action given by $(e^{i\phi_1}, e^{i\phi_2}) \cdot (e^{i\theta_1}, e^{i\theta_2}, e^{i\psi_1}, e^{i\psi_2}) := (e^{i(\theta_1+\phi_1)}, e^{i(\theta_2+\phi_2)}, e^{i(\psi_1+\phi_1)}, e^{i(\psi_2+\phi_2)})$. First of all, notice that since the two-torus is compact this action is necessarily proper. Moreover, as \mathbb{T}^2 acts freely, the corresponding orbit space $M/A_{\mathbb{T}^2}$ is a smooth manifold such that the projection $\pi_{A_{\mathbb{T}^2}} : M \rightarrow M/A_{\mathbb{T}^2}$ is a surjective submersion. The polar distribution $A'_{\mathbb{T}^2}$ does not have that property. Indeed, $C^\infty(M)^{\mathbb{T}^2}$ comprises all the functions f of the form $f \equiv f(e^{i(\theta_1-\psi_1)}, e^{i(\theta_2-\psi_2)})$. An inspection of the Hamiltonian flows associated to such functions readily shows that the leaves of $A'_{\mathbb{T}^2}$, that is, the level sets of the optimal momentum map \mathcal{J} , are the products of two leaves of an irrational foliation in a two-torus. Moreover, it can be checked that for any $\rho \in M/A_{\mathbb{T}^2}$, the isotropy subgroup \mathbb{T}_ρ^2 is the product of two discrete subgroups of S^1 , each of which fill densely the circle. We can use this density property to show that the \mathbb{T}_ρ -action on $\mathcal{J}^{-1}(\rho)$ is not proper. Let $\{(e^{i\tau_n}, e^{i\sigma_n})\}_{n \in \mathbb{N}}$ be a strictly monotone sequence of elements in \mathbb{T}_ρ^2 that converges to (e, e) in \mathbb{T}^2 . Then, for any sequence $\{z_n\}_{n \in \mathbb{N}} \subset \mathcal{J}^{-1}(\rho)$ such that $z_n \rightarrow z \in \mathcal{J}^{-1}(\rho)$ in $\mathcal{J}^{-1}(\rho)$ we have that $(e^{i\tau_n}, e^{i\sigma_n}) \cdot z_n \rightarrow z$ in $\mathcal{J}^{-1}(\rho)$. However, since \mathbb{T}_ρ^2 is endowed with the discrete topology and $\{(e^{i\tau_n}, e^{i\sigma_n})\}_{n \in \mathbb{N}}$ is strictly monotone it has no convergent subsequences, which implies that G_ρ does not act properly on $\mathcal{J}^{-1}(\rho)$.

Example 2. – A simplified version of the previous example provides a situation where the hypotheses of Theorem 3.1 are satisfied while all the standard reduction theorems fail. Namely, there are no momentum maps for this action and, moreover, the “closedness hypothesis” in [12] is not satisfied.

Let $M := \mathbb{T}^2 \times \mathbb{T}^2$ with the same symplectic structure that we had in the previous example. We now consider the canonical circle action given by $e^{i\phi} \cdot (e^{i\theta_1}, e^{i\theta_2}, e^{i\psi_1}, e^{i\psi_2}) := (e^{i(\theta_1+\phi)}, e^{i\theta_2}, e^{i(\psi_1+\phi)}, e^{i\psi_2})$. In this case, $C^\infty(M)^{S^1}$ comprises all the functions f of the form $f \equiv f(e^{i\theta_2}, e^{i\psi_2}, e^{i(\theta_1-\psi_1)})$. An inspection of the Hamiltonian flows associated to such functions readily shows that the leaves of A'_{S^1} , that is, the level sets $\mathcal{J}^{-1}(\rho)$ of the optimal momentum map \mathcal{J} , are the product of a two-torus with a leaf of an irrational foliation (Kronecker submanifold) of another two-torus. Obviously this is not compatible with the existence of a $(\mathbb{R}^2$ or \mathbb{T}^2 -valued) momentum map or with the closedness hypothesis in [12]. Nevertheless,

the isotropies S_ρ^1 coincide with the circle S^1 , whose compactness guarantees that its action on $\mathcal{J}^{-1}(\rho)$ is proper. Theorem 3.1 automatically guarantees that the quotients of the form $M_\rho := \mathcal{J}^{-1}(\rho)/S_\rho^1 \simeq (S^1 \times_{S^1} S^1) \times \{\text{Kronecker submanifold of } \mathbb{T}^2\}$ are symplectic.

Proof of the theorem. – Since by hypothesis the G_ρ -action on $\mathcal{J}^{-1}(\rho)$ is proper and by Proposition 2.2 it has fixed isotropies, the quotient $\mathcal{J}^{-1}(\rho)/G_\rho$ is therefore a smooth manifold, and the projection $\pi_\rho : \mathcal{J}^{-1}(\rho) \rightarrow \mathcal{J}^{-1}(\rho)/G_\rho$ is a smooth surjective submersion.

We start the proof of the symplecticity of M_ρ by showing that (1) is a good definition for the form ω_ρ in the quotient M_ρ . Let $m, m' \in \mathcal{J}^{-1}(\rho)$ be such that $\pi_\rho(m) = \pi_\rho(m')$, and $v, w \in T_m\mathcal{J}^{-1}(\rho)$, $v', w' \in T_{m'}\mathcal{J}^{-1}(\rho)$ be such that $T_m\pi_\rho \cdot v = T_{m'}\pi_\rho \cdot v'$, $T_m\pi_\rho \cdot w = T_{m'}\pi_\rho \cdot w'$. Let $f, f', g, g' \in C^\infty(M)^G$ be such that $v = X_f(m)$, $v' = X_{f'}(m')$, $w = X_g(m)$, $w' = X_{g'}(m')$. The condition $\pi_\rho(m) = \pi_\rho(m')$ implies the existence of an element $k \in G_\rho$ such that $m' = \Phi_k^\rho(m)$. We also have that $T_m\pi_\rho = T_{m'}\pi_\rho \circ T_m\Phi_k^\rho$. Analogously, because of the equalities $T_m\pi_\rho \cdot v = T_{m'}\pi_\rho \cdot v'$, $T_m\pi_\rho \cdot w = T_{m'}\pi_\rho \cdot w'$ there exist G -invariant functions $h^1, h^2 \in C^\infty(M)^G$ and elements $\xi^1, \xi^2 \in \mathfrak{g}_\rho$ such that $X_{f'}(m') - T_m\Phi_k^\rho \cdot X_f(m) = \xi_{\mathcal{J}^{-1}(\rho)}^1(m') = X_{h^1}(m')$, and $X_{g'}(m') - T_m\Phi_k^\rho \cdot X_g(m) = \xi_{\mathcal{J}^{-1}(\rho)}^2(m') = X_{h^2}(m')$, or, analogously $X_{f'}(m') = X_{h^1+f \circ \Phi_{k^{-1}}}(m') = X_{h^1+f}(m')$, and $X_{g'}(m') = X_{h^2+g \circ \Phi_{k^{-1}}}(m') = X_{h^2+g}(m')$. Hence, we can write

$$\begin{aligned} \omega_\rho(\pi_\rho(m'))(v', w') &= \{f', g'\}(m') = \{h^1 + f, h^2 + g\}(m') = \{h^1 + f, h^2 + g\}(m) \\ &= \{f, g\}(m) + \{f, h^2\}(m) + \{h^1, g\}(m) + \{h^1, h^2\}(m) \\ &= \{f, g\}(m) + \mathbf{d}f(m) \cdot \xi_{\mathcal{J}^{-1}(\rho)}^2(m) - \mathbf{d}(g + h^2)(m) \cdot \xi_{\mathcal{J}^{-1}(\rho)}^1(m) \\ &= \{f, g\}(m) = \omega_\rho(\pi_\rho(m))(v, w). \end{aligned}$$

Consequently, ω_ρ is a well defined two-form on the quotient M_ρ . Given that π_ρ is a smooth surjective submersion, the form ω_ρ is clearly smooth. The Jacobi identity for the bracket $\{\cdot, \cdot\}$ on M implies that ω_ρ is closed. These two features of the form ω_ρ can also be immediately read out of the expression for ω_ρ given in Remark 1, whose equivalence with (1) is straightforward.

It only remains to be shown that ω_ρ is non degenerate. We start our argument with a few notations and remarks. Let $H \subset G$ be the isotropy subgroup of all the elements in $\mathcal{J}^{-1}(\rho)$ with respect to the smooth G_ρ -action on this manifold. Recall that by Proposition 2.2 this isotropy subgroup coincides with an isotropy of the G -action on M . Since by hypothesis the G -action on M is proper, the subgroup $H \subset G_\rho$ is necessarily compact. Moreover, the Slice theorem guarantees that for any point $m \in \mathcal{J}^{-1}(\rho)$, there is a G -invariant neighborhood U of m in M that is G -equivariantly diffeomorphic to the twist product $G \times_H V_r$, where V_r is a ball of radius r around the origin in some vector space V on which H acts linearly.

Let $m \in \mathcal{J}^{-1}(\rho)$ arbitrary. Suppose that the vector $X_f(m)$, with $f \in C^\infty(M)^G$, satisfies the equality $\pi_\rho^*\omega_\rho(m)(X_f(m), X_h(m)) = \{f, h\}(m) = 0$, for all $h \in C^\infty(M)^G$. In order to prove that ω_ρ is non degenerate we have to show that $X_f(m) \in T_m(G_\rho \cdot m)$. We will do so by using the local coordinates around the point m provided by the Slice theorem. First of all, as f is G -invariant $X_f(m) \in T_mM_H$. Hence, as in local coordinates $M_H \simeq N(H) \times_H V_r^H$, we have that $X_f(m) = T_{(e,0)}\pi \cdot (\zeta, v)$, where $\pi : G \times V_r \rightarrow G \times_H V_r$ is the natural projection, $\zeta \in \text{Lie}(N(H))$, and $v \in V^H$. We recall that V^H denotes the fixed points in V by the action of H .

We now rephrase in these local coordinates the non degeneracy condition. Indeed, the fact that $\pi_\rho^*\omega_\rho(m)(X_f(m), X_h(m)) = \{f, h\}(m) = -\mathbf{d}h(m) \cdot X_f(m) = 0$, for all $h \in C^\infty(M)^G$ amounts to saying that $\mathbf{d}g(0) \cdot v = 0$ for all the functions $g \in C^\infty(V_r)^H$. On other words, $v \in (\{\mathbf{d}g(0) \mid g \in C^\infty(V_r)^H\})^\circ$. A known fact about proper group actions (see Proposition 3.1.1 in [10] or Proposition 2.14 in [12]) implies that $v \in ((V^*)^H)^\circ$. Consequently, $v \in V^H \cap ((V^*)^H)^\circ$. We now show that this intersection is trivial and therefore $v = 0$ necessarily.

We start by recalling (*see* again the references that we just quoted) that the restriction to $(V^*)^H$ of the dual map associated to the inclusion $i_{V^H} : V^H \hookrightarrow V$ is a H -equivariant isomorphism from $(V^*)^H$ to $(V^H)^*$. Now, as $v \in V^H \cap ((V^*)^H)^\circ$ we have that $\langle \alpha, v \rangle_V = 0$ for every $\alpha \in ((V^*)^H)$. The symbol $\langle \cdot, \cdot \rangle_V$ denotes the natural pairing of V with its dual. We can rewrite this condition as $0 = \langle \alpha, v \rangle_V = \langle \alpha, i_{V^H}(v) \rangle_V = \langle i_{V^H}^*(\alpha), v \rangle_{V^H}$. As the restriction $i_{V^H}^*|_{(V^*)^H}$ is an isomorphism, the previous identity is equivalent to $\langle \beta, v \rangle_{V^H} = 0$ for all $\beta \in (V^H)^*$. Consequently, $v = 0$, as required.

We conclude our argument by noting that as $X_f(m) = T_{(e,0)}\pi \cdot (\zeta, 0)$, we have that $X_f(m) \in T_m(G \cdot m) \cap A'_G(m) = T_m(G_\rho \cdot m)$, which proves the nondegeneracy of ω_ρ . \square

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