Critical point theory and Hamiltonian dynamics around critical elements

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Abstract

We give a few examples of how, using the knowledge available on the geometry of Hamiltonian dynamics with symmetry, standard critical point theory can be adapted to this setup in order to obtain predictions on the existence of various dynamical elements and, moreover, it can be used to provide estimates on the number of these solutions. The proofs of these results, as well as additional information, can be found in [OR00, O00]

1 Critical point theory in the Hamiltonian symmetric context

The use of the symmetries of a mechanical system, propitiated by all the founders of the field (especially H. Poincaré and E. Nöther) has produced very important fruits. In our discussion we will significantly use the philosophy advocated by S. Smale [S70] who proposed the study of the topology of the spaces resulting by quotienting the level sets of the existing conserved quantities (energy and momentum) by the relevant group actions. As we will see, a thorough study of this reduced kinematics is capable to produce valuable dynamical information.

To be more specific, we will look for relative critical elements of a symmetric Hamiltonian system by reducing their search to the existence of critical points of an appropriate function to one of the reduced spaces proposed by Smale. Once we have formulated the problem in such a way we will use some methods of critical point theory, mainly Lusternik-Schnirelman and Morse theories, in order to solve it.

As to the Lusternik-Schnirelman categorical approach we will use mainly three ideas:

- Let M be a compact G–manifold, with G a compact Lie group. Any G–invariant smooth function $f \in C^\infty(M)^G$ has at least $\mathrm{Cat}(M/G)$ critical orbits [W77].

- Let M be a compact G–manifold, with G a Lie group acting properly on M such that the isotropy subgroup of each point $m \in M$ is a finite subgroup of G. Let ω be a symplectic G–invariant form defined on M. Let $H \subset G$ be a Lie subgroup of G and $N \subset M$ be a H–invariant closed submanifold of M such that for any $n \in N$ we have that

$$(T_n N)^\omega = \mathfrak{g} \cdot n \qquad \text{and} \qquad T_n N \cap \mathfrak{g} \cdot m = \mathfrak{h} \cdot n, \tag{1.1}$$

  where $\mathfrak{g} \cdot n = \left\{ \xi_M(n) := \frac{d}{dt}\big|_{t=0} \exp t\xi \cdot n \mid \xi \in \mathfrak{g} \right\}$ denotes the tangent space at $n \in N \subset M$ of the G–orbit $G \cdot n = \{g \cdot n \mid g \in G\}$ and $(T_n N)^\omega$ denotes the ω–orthogonal space of $T_n N$ in $T_n M$. Then, there is a cohomology class $\theta \in H^2(N/H; \mathbb{R})$ such that $\theta^k \neq 0$, where $k = \frac{1}{2}(\dim N - \dim H)$ [W77]. This property amounts to a lower bound for the value of the Lusternik-Schnirelman category of a symplectic quotient [MW74] resulting from a locally free action.

- Let G be a compact Lie group that contains a maximal torus T and that acts linearly on the vector space V. Suppose that the vector subspace V^T of T–fixed vectors on V is trivial, that is, $V^T = \{0\}$, then any G–invariant function defined in the unit sphere of V (the unit sphere of V is defined with the aid of a G–invariant norm on it) has at least

$$\frac{\dim V}{2(1 + \dim G - \dim T)} = \frac{\dim V}{2(1 + \dim G - \operatorname{rank} G)}$$

  critical orbits [Ba94].

Regarding Morse theory, the symplectically reduced spaces where we will search for critical points are exactly those studied by Kirwan in [KW84], hence we will be able to use in a straightforward manner her results.

2 Relative equilibria and periodic orbits around non degenerate critical points

We illustrate the techniques discussed in the previous section with a simple application whose proof can be found in [OR00].

Theorem 2.1 *Let $(V, \omega, h, G, \mathbf{J})$ be a Hamiltonian G–vector space, with G a compact Lie group. Suppose that $h(0) = 0$, $\mathbf{d}h(0) = 0$, and the quadratic form $Q := \mathbf{d}^2 h(0)$ on V is definite. Let $\xi \in \mathfrak{g}$ be such that the quadratic form $\mathbf{d}^2 \mathbf{J}^\xi(0)$ is non degenerate. Then, for each energy value ϵ small enough, there are at least*

$$\operatorname{Cat}\left(h^{-1}(\epsilon)/G^\xi\right) = \operatorname{Cat}\left(Q^{-1}(\epsilon)/G^\xi\right) \tag{2.1}$$

distinct relative equilibria in $h^{-1}(\epsilon)$ whose velocities are (real) multiples of ξ. The symbol $G^\xi := \{g \in G \mid \operatorname{Ad}_g \xi = \xi\}$ denotes the adjoint isotropy of the element $\xi \in \mathfrak{g}$ and Cat *is the*

Lusternik–Schnirelman category. If the compact Lie group G^ξ has a maximal torus T^ξ such that the set V^{T^ξ} of T^ξ–fixed vectors on V is trivial, that is, $V^{T^\xi} = \{0\}$, then there are at least

$$\frac{\dim V}{2(1 + \dim G^\xi - \operatorname{rank} G^\xi)}. \tag{2.2}$$

distinct relative equilibria in $h^{-1}(\epsilon)$ whose velocities are (real) multiples of ξ.

If the G-symmetry in the preceeding theorem is given by the S^1-action resulting of putting the system in normal form [VvdM95] we obtain as a corollary a generalization to the symmetric framework of the Weinstein-Moser Theorem on the existence of periodic orbits around a stable equilibrium. More explicitely, we have [MRS88, Ba94, OR00]:

Corollary 2.2 (Equivariant Weinstein–Moser theorem) *Let $(V, \omega, h, G, \mathbf{J})$ be a Hamiltonian G–vector space, with G a compact Lie group, such that $h(0) = 0$, $\mathbf{d}h(0) = 0$, and the infinitesimally symplectic linear map $A := DX_h(0)$ is non singular and has $\pm i\nu_\circ$ in its spectrum. Let $U_{\nu_\circ}$ be the resonance space of A with primitive period $T_{\nu_\circ}$ and consider the canonical S^1–action on $U_{\nu_\circ}$ defined by the restriction to this symplectic subspace of the flow of A. Let $H \subset G \times S^1$ be an isotropy subgroup of the $G \times S^1$–action on $U_{\nu_\circ}$. If the restriction of $Q := \mathbf{d}^2 h(0)$ to $(U_{\nu_\circ})^H$ is a definite quadratic form then, for any sufficiently small $\epsilon > 0$ there are at least*

$$\frac{\dim (U_{\nu_\circ})^H}{2(1 + \dim (N(H)/H) - \operatorname{rank}(\mathrm{N(H)/H}))} \tag{2.3}$$

geometrically distinct periodic orbits in each energy level $h^{-1}(\epsilon)$ whose periods tend to $T_{\nu_\circ}$ as ϵ tends to zero and whose isotropy subgroups include H.

3 Relative periodic orbits

This section is a brief account of the main result in [O00]. A genuine generalization of the Weinstein-Moser Theorem in the symmetric framework should be able to predict *relative periodic orbits* (RPOs) in the neighboring energy-momentum level sets of a stable *relative equilibrium*. Moreover, it should be able to give an estimate on the number of these solutions for any prescribed spatiotemporal structure, that is, for any isotropy subgroup of the natural $G \times S^1$-action present in the problem. In this direction we have:

Theorem 3.1 *Let $(V, \omega, h, G, \mathbf{J} : V \to \mathfrak{g}^*)$ be a Hamiltonian system with symmetry, with V a vector space, and G a compact positive dimensional Lie group that acts on V in a linear and canonical fashion. Suppose that $h(0) = 0$, $\mathbf{d}h(0) = 0$ (that is, the Hamiltonian vector field X_h has an equilibrium at the origin) and that the linear Hamiltonian vector field $A := DX_h(0)$ is non degenerate and contains $\pm i\nu_\circ$ in its spectrum. Let $U_{\nu_\circ}$ be the resonance space of A with primitive period $T_{\nu_\circ} := \frac{2\pi}{\nu_\circ}$. Consider the $G \times S^1$–action on $U_{\nu_\circ}$, where the S^1–action is induced* *by the semisimple part of A, and the Lie group G acts simply on $U_{\nu_\circ}$ Let $H = \{(k, \theta_H(k)) \mid k \in K \subset G\} \subset G \times S^1$ be an isotropy subgroup of the $G \times S^1$–action on $U_{\nu_\circ}$ with temporal character θ_H, temporal velocity $\rho_H \in \mathfrak{k}^*$, and such that the quadratic form Q^H on the H–fixed point space $U^H_{\nu_\circ}$ defined by*

$$Q^H(v) := \frac{1}{2}\mathbf{d}^2 h(0)(v,v), \qquad v \in U^H_{\nu_\circ}$$

is definite. Then, for any $\chi_\circ \in (\mathfrak{k}^\circ)^K$ for which $\mathbf{J}|^{-1}_{(U_{\nu_\circ})_H}\left(\chi_\circ - \frac{1}{\nu_\circ}\rho_H\right) \cap Q_H^{-1}(1)$ is non empty ($Q_H := Q^H|_{(U_{\nu_\circ})_H}$) there exists an open neighborhood $V_{\chi_\circ}$ of $\chi_\circ$ in $(\mathfrak{k}^\circ)^K$ such that for any $\chi \in V_{\chi_\circ}$, the intersection $\mathbf{J}|^{-1}_{(U_{\nu_\circ})_H}\left(\chi - \frac{1}{\nu_\circ}\rho_H\right) \cap Q_H^{-1}(1)$ is a submanifold of $(U_{\nu_\circ})_H$ of dimension $\dim U^H_{\nu_\circ} - \dim N(H)/H$. *Suppose that the following two generic hypotheses hold:*

(H1) *The restriction $h|_{U^H_{\nu_\circ}}$ of the Hamiltonian h to the fixed point subspace $U^H_{\nu_\circ}$ is not radial with respect to the norm associated to Q^H.*

(H2) *Let $h_k(v) := \frac{1}{k!}\mathbf{d}^k h(0)\left(v^{(k)}\right)$, $v \in U^H_{\nu_\circ}$ be the first non radial term in the Taylor expansion of $h|_{U^H_{\nu_\circ}}$ around zero. We will assume that $k \geq 4$ and that the restrictions of h_k to the submanifolds $\mathbf{J}|^{-1}_{(U_{\nu_\circ})_H}\left(\chi - \frac{1}{\nu_\circ}\rho_H\right) \cap Q_H^{-1}(1)$, with $\chi \in V_{\chi_\circ}$, are Morse–Bott functions with respect to the $(N_G(K)_{\rho_H} \cap N_G(K)_\chi) \times S^1$–action.*

Then, for any $\epsilon > 0$ close enough to zero, $\chi \in V_{\chi_\circ}$, and $\lambda := \Xi^*(\chi - \frac{1}{\nu_\circ}\rho_H, \frac{1}{\nu_\circ})$, there are at least

$$\max\left[\frac{1}{2}\left(\dim U^H_{\nu_\circ} - \dim N_G(K) - \dim\left(N_G(K)_{\rho_H} \cap N_G(K)_\chi\right) + 2\dim K\right), \chi_E\left(\mathbf{EJ}^{-1}_{L_H}(\lambda)\right)^{L_\lambda}\right] \tag{3.1}$$

distinct relative periodic orbits of X_h with energy ϵ, momentum $\epsilon(\chi - \frac{1}{\nu_\circ}\rho_H) \in \mathfrak{g}^*$, isotropy subgroup H, and relative period close to $T_{\nu_\circ}$. The symbol $\chi_E\left(\mathbf{EJ}^{-1}_{L_H}(\lambda)\right)^{L_\lambda}$ denotes the L_λ–Euler characteristic of $\mathbf{EJ}^{-1}_{L_H}(\lambda)$ (which in this case equals the standard Euler characteristic of the symplectic quotient $\chi_E(\mathbf{EJ}^{-1}_{L_H}(\lambda)/L_\lambda)$.

The first part of estimate 3.1 is a product of the use of the Lusternik-Schnirelman category on the problem, while the second one is obviously coming out of symplectic Morse theory.

4 Further generalizations and future directions

For the sake of simplicity, in the preceeding two sections we chose to present results that predict periodic and relative periodic solutions, as well as relative equilibria, in the neighborhood of a symmetric *equilibrium on a symplectic vector space*. This situation can be generalized to *relative* *equilibria on a manifold* using the so called *reconstruction equations* [O98, RWL99, OR01] on the normal coordinates of Marle, Guillemin, and Sternberg [Mar84, Mar85, GS84].

All the situations considered in the statements presented above involve non degeneracy hypotheses of one form or another. These restrictions allow to preserve the Hamiltonian structure through all the reductions carried out to solve the problem. The degenerate situations involve completely different techniques and require a much more analytic study. In other words, the geometry of the problem is still important but its understanding is not sufficient to fully describe the dynamics associated to it. The reader is encouraged to check with [OR00] for results similar to the ones presented here, in the presence of degeneracies.

The previous statement is especially relevant when attempting to understand the bifurcation phenomena in Hamiltonian symmetric systems. For instance, the appearance of RPOs in a Hamiltonian symmetric Hopf bifurcation is a natural problem to study where some answers have been given [COR99]. Nevertheless, the full picture is far from being understood.

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