SOME REMARKS ABOUT THE GEOMETRY OF HAMILTONIAN CONSERVATION LAWS

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We briefly review some results on distribution and singular foliation theory applied to the symmetric Hamiltonian framework and we see how this approach provides a unified picture of the various reduction schemes that can be found in the literature dealing with both conservative and dissipative systems.

1. Introduction

The use of symmetries in the quantitative and qualitative study of dynamical systems has a long history that goes back to the founders of mechanics and that we will not attempt to review here.

In most cases, the symmetries of a system are used in carrying out a procedure generically known under the name of ***reduction*** that restricts the study of its dynamics to a dimensionally smaller system. In the dissipative framework (see for instance Golubitsky *et al* [9], Chossat and Lauterbach [5], Golubitsky and Stewart [10], and references therein) this procedure is usually implemented by looking at the ***isotropy type submanifolds*** associated the Lie group action action that encodes the symmetry of the problem and modding out by the relevant residual group action (we will briefly review this process below). In the Hamiltonian setup the reduction technology is usually linked to the presence of a ***momentum map*** whose level sets are preserved by G–equivariant dynamics via Noether's Theorem. In this context, the reduction procedure is usually referred to as ***symplectic or Marsden–Weinstein reduction*** [20].

In principle, these two approaches are not related to each other in the sense that they are based on different conservation laws. On other words, the momentum map does not see the isotropy type submanifolds and viceversa. In the following pages we will briefly review a procedure based on distribution theory that in the Hamiltonian context unifies these two approaches and generalizes the existing results to the category of Poisson manifolds under hypotheses that do not necessarily imply the existence of a standard momentum map.

2. The dissipative case

Let M be a smooth manifold and G be a Lie group acting properly on M. Let $X \in \mathfrak{X}(M)^G$ be a G–equivariant vector field on M and F_t be the corresponding (also equivariant) flow. For any isotropy subgroup H of the G–action on M, the H–***isotropy type submanifold*** M_H defined by

$$M_H := \{m \in M \mid G_m = H\} \tag{1}$$

is preserved by the flow F_t. The symbol G_m denotes the isotropy subgroup of the element $m \in M$. The proper character of the action guarantees that (the connected components of) M_H is an actual submanifold of M that, in general, is not closed. Moreover, the quotient group $N(H)/H$ (where $N(H)$ denotes the normalizer of H in G) acts freely and effectively on M_H. Hence, if $\pi_H : M_H \to M_H/(N(H)/H)$ denotes the projection onto orbit space and $i_H : M_H \hookrightarrow M$ the injection, the vector field X induces a unique vector field X^H on the quotient $M_H/(N(H)/H)$ via the expression:

$$X^H \circ \pi_H = T\pi_H \circ X \circ i_H,$$

whose flow F_t^H is given by $F_t^H \circ \pi_H = \pi_H \circ F_t \circ i_H$. We will refer to $X^H \in \mathfrak{X}\left(M_H/(N(H)/H)\right)$ as the H–***isotropy type reduced vector field*** corresponding to X.

This reduction technique has been profusely exploited in specific examples (see [9], [5], and [10]). When the symmetry group G is compact and we are dealing with a linear action the construction of the quotient $M_H/(N(H)/H)$ can be implemented in a very explicit and convenient manner by using the invariant polynomials of the action and the theorems of Hilbert and Schwarz–Mather. Apart from the already cited works, the reader may want to check with [11] [6] [12] [13] regarding this point.

3. The Hamiltonian case

Let (M, ω) be a symplectic manifold and G be a compact connected Lie group acting freely on (M, ω) by symplectomorphisms with Lie algebra $\mathfrak{g}$. Suppose that this action has a standard equivariant momentum map $\mathbf{J} : M \to \mathfrak{g}^*$ defined by the condition that for any element $\xi \in \mathfrak{g}$, the Hamiltonian vector field $X_{\mathbf{J}^\xi}$ associated to the function $\mathbf{J}^\xi := \langle \mathbf{J}, \xi \rangle$ is such that $X_{\mathbf{J}^\xi} = \xi_M$, with ξ_M the infinitesimal generator vector field associated to $\xi \in \mathfrak{g}$.

The Marsden–Weinstein reduction theorem [20] says that for any $\mu \in \mathbf{J}(M) \subset \mathfrak{g}^*$, the quotient $M_\mu := \mathbf{J}^{-1}(\mu)/G_\mu$ is a symplectic manifold with symplectic form ω_μ uniquely determined by the equality $\pi_\mu^* \omega_\mu = i_\mu^* \omega$, where G_μ is the isotropy subgroup of the element $\mu \in \mathfrak{g}^*$ with respect to the coadjoint action of G on $\mathfrak{g}^*$, $i_\mu : \mathbf{J}^{-1}(\mu) \hookrightarrow M$ is the canonical injection, and $\pi_\mu : \mathbf{J}^{-1}(\mu) \to \mathbf{J}^{-1}(\mu)/G_\mu$ the projection onto the orbit space.

In terms of dynamics, the interest of this construction is given by the fact that for any G–invariant Hamiltonian $h \in C^\infty(M)^G$, the corresponding Hamiltonian flow F_t leaves invariant the connected components of $\mathbf{J}^{-1}(\mu)$ (***Noether's Theorem***) and commutes with the G–action, so it induces a flow F_t^μ on M_μ, uniquely determined by $\pi_\mu \circ F_t \circ i_\mu = F_t^\mu \circ \pi_\mu$. The flow F_t^μ is Hamiltonian in (M_μ, ω_μ), with Hamiltonian function $h_\mu \in C^\infty(M_\mu)$ defined by the relation $h_\mu \circ \pi_\mu = h \circ i_\mu$. The function h_μ is called the ***reduced Hamiltonian***.

Symplectic reduction is a very powerful tool that has been involved in countless developments in symplectic geometry and in the study of Hamiltonian dynamical systems with symmetry [1]. Nevertheless, there are situations in which the procedure that we just described does not work or is not efficient enough. For instance, we could encounter the following situations:

- The symmetry of the system does not have a momentum map associated. This problem has been solved in some situations with the introduction of other momentum maps [7] [21] [2].
- The action is not free and therefore the symplectic quotient M_μ is not a smooth manifold. In the presence of a momentum map this situation has been treated in [30] [4] [23] [8].
- The symmetry group is discrete and therefore the momentum map does not provide any conservation law.
- The phase space of our system is not a symplectic but a Poisson manifold [19] [27].

4. The distribution theoretical approach

In this section we introduce an approach to the conservation laws in symmetric systems that unifies the procedures presented for the dissipative and Hamiltonian cases and overcomes the difficulties in the conservative case enumerated at the end of the previous section.

We start with the case of general G–equivariant dynamical systems where we will see that the isotropy type submanifolds and the associated reduced spaces can be obtained out of a very natural construction involving just the basic ingredients of this setup. Let $\mathfrak{X}(M)^G$ be the set of G–equivariant smooth vector fields on M and D be the smooth generalized distribution [31] [32] [33] on

M defined by

$$D(m) := \operatorname{span}\{X(m) \mid X \in \mathfrak{X}(M)^G\}, \quad m \in M. \tag{2}$$

A straightforward computation shows that D is integrable in the sense of Stefan [31], [32] and Sussman [33]. A slightly more advanced argument using the Slice Theorem proves that the accessible sets or integrable manifolds of this distribution are actually the connected components of the isotropy type manifolds M_H defined in (1). In order to keep the notation manageable we will assume in the sequel that the isotropy type manifolds M_H are connected. Let M/D be the corresponding leaf space and $\mathcal{J} : M \to M/D$ be the projection. The very definition of the distribution D allows us to define a (in general not smooth) G–action on the quotient M/D by $g \cdot \mathcal{J}(m) := \mathcal{J}(g \cdot m)$, for all $m \in M$. It can be checked that the isotropy subgroup G_ρ of an element $\rho := \mathcal{J}(m) \in M/D$ coincides with the normalizer $N(G_m)$ of G_m in G. Consequently, if we take an element $m \in M$ such that $\mathcal{J}(m) = \rho \in M/D$ and whose isotropy subgroup G_m equals H, the reduced space $M_H/(N(H)/G)$ introduced in Section 2 can be rewritten as $\mathcal{J}^{-1}(\rho)/G_\rho$, an expression that strongly resembles the Marsden–Weinstein symplectic reduced space, even though in principle it has nothing to do with it.

What we now show in the following paragraphs is that this construction is, from the distribution theoretical perspective that we adopted in this section, much related to the Marsden–Weinstein reduction procedure. More specifically, if we replace the previous definitions by their analogues in the category presented in Section 3 we obtain exactly the Marsden–Weinstein reduced spaces. However, the class of examples in which the definitions that we will introduce makes sense goes beyond that category, that is, free actions on a symplectic manifold that have an equivariant momentum map. Indeed, the natural setup for those ideas is the category of Poisson manifolds endowed with a canonical action. The following paragraphs are a brief description of the ***optimal momentum map*** and the reduction that can be carried out with it, as it has been described in [28] [25] [26] [29].

Let $(M, \{\cdot,\cdot\})$ be a Poisson manifold and G be a Lie group acting properly and canonically on M. We now consider the distribution, we call it A'_G, that naturally adapts the distribution D in (2) to this setup:

$$A'_G(m) := \{X_f(m) \mid f \in C^\infty(M)^G\}, \quad \text{for all} \quad m \in M.$$

The symbol X_f denotes the equivariant Hamiltonian vector field associated to the invariant function $f \in C^\infty(M)^G$. This distribution is also integrable and therefore has an associated leaf space M/A'_G where a natural G–action can be defined. We will refer to the projection $\mathcal{J} : M \to M/A'_G$ as the

optimal momentum map. This denomination is justified by the fact that, by construction, the fibers of this map are preserved by the flows of all the G–equivariant Hamiltonian vector fields on M (as the standard momentum map does) and these fibers are the smallest (initial) submanifolds preserved by all those flows (it is optimal). It can also be proved that $\mathcal{J}$ is universal in a certain sense [28].

As far as reduction is concerned, the main use of the optimal momentum map is the following theorem [25]:

Theorem 4.1. *Let* $(M, \{\cdot,\cdot\})$ *be a smooth Poisson manifold and* G *be a Lie group acting canonically and properly on* M*. Let* $\mathcal{J} : M \to M/A'_G$ *be the optimal momentum map associated to this action. Then, for any* $\rho \in M/A'_G$ *whose isotropy subgroup* G_ρ *acts properly on* $\mathcal{J}^{-1}(\rho)$*, the orbit space* $M_\rho := \mathcal{J}^{-1}(\rho)/G_\rho$ *is a smooth symplectic regular quotient manifold with symplectic form* ω_ρ *defined by*

$$\pi_\rho^*\omega_\rho(m)(X_f(m), X_h(m)) = \{f, h\}(m),$$

for any $m \in \mathcal{J}^{-1}(\rho)$ *and* $f, h \in C^\infty(M)^G$*. The map* $\pi_\rho : \mathcal{J}^{-1}(\rho) \to \mathcal{J}^{-1}(\rho)/G_\rho$ *is the canonical projection onto the orbit space of the* G_ρ*–action on* $\mathcal{J}^{-1}(\rho)$*. We will refer to the pair* (M_ρ, ω_ρ) *as the* ***(optimal) point reduced space*** *of* $(M, \{\cdot,\cdot\})$ *at* ρ*.*

This theorem shows that the optimal or distribution theoretical approach to Hamiltonian reduction solves some of the inconveniences of the traditional approach mentioned at the end of Section 3. First of all notice that the hypotheses of this result do not require the presence of a standard momentum map in the picture and that the theorem is general enough to comprise the Poisson case. Moreover, there are no assumptions on the freeness of the action and, as we will see below, the theorem still provides valuable information when the symmetry group is discrete.

In order to make the content of this result more visible, we describe the optimal reduced spaces in some situations where they can be expressed in terms of familiar objects. The proofs of the following statements can be found in [28], [26]. Suppose that M is a symplectic manifold. Then:

- If there is a Lie group acting freely, properly, canonically, and this action has a momentum map associated then the optimal reduced spaces coincide (up to connected components) with the Marsden–Weinstein reduced spaces.
- If in the previous setup we drop the freeness hypothesis, the optimal reduced spaces coincide with the singular reduced spaces of Sjamaar and Lerman [30] [4] [23] [29].

- If the group is discrete we obtain as reduced spaces the quotient spaces $M_H/(N(H)/H)$ that, by our theorem, happen to be symplectic.

These statements clearly show that the distribution approach provides a uniform description of several reduction techniques that appear in the literature, adapted to different sets of hypotheses, both in the dissipative and in the Hamiltonian case and that at first sight looked unrelated to each other.

5. Another related developments

Singular dual pairs. It can be proved that when (M, ω) is a symplectic manifold and G is a Lie group acting canonically and properly on M, then the distribution A'_G is such that

$$A'_G(m) = (\mathfrak{g} \cdot m)^\omega \cap T_m M_{G_m}, \quad \text{for all} \quad m \in M,$$

where $(\mathfrak{g} \cdot m)^\omega$ is the symplectic orthogonal vector space to the tangent space $\mathfrak{g} \cdot m$ to the G–orbit. When the action is free and the isotropy subgroups $G_m = \{e\}$ for all $m \in M$, then $A'_G(m)$ coincides with $(\mathfrak{g} \cdot m)^\omega$. Consequently, A'_G can be thought of as a generalization to the singular (non free) case of the notion of symplectic orthogonality (polarity). This idea has been exploited in [24] in the construction of a theory of singular dual pairs that generalizes to the non regular context the classical ideas of Lie [14] and Weinstein [34].

Orbit reduction and homogeneous presymplectic manifolds. The Marsden–Weinstein reduction scheme has a close relative where the dynamically invariant manifolds that are considered are G–invariant. This approach involves the Lie–Poisson structure in the dual of the Lie algebra of the symmetry group as well as the Kostant–Kirillov–Souriau symplectic structures of the corresponding coadjoint orbits. This procedure, usually known as orbit reduction, can be reproduced in the optimal context. In this case the coadjoint orbits are replaced by presymplectic homogeneous manifolds that admit a natural symplectic decomposition. The reader can check the details in [26].

Reduction by stages. Whenever the symmetry group G has a normal subgroup N, the Marsden–Weinstein reduction scheme can be implemented in two stages. A first stage involves the symplectic reduction by N which is followed by a reduction by the quotient group G/N. The Reduction by Stages Theorem [17] [18] states that when certain hypotheses are met, the resulting reduced spaces coincide with the "one–shot" reduced space. This result, very important in practical applications, can be reproduced in the optimal context [26].

6. Future directions and work in progress

Nonholonomic symmetric mechanics. The distribution theoretical component of the ideas presented in the preceding pages makes them specially suitable in the treatment of symmetric systems with nonholonomic constraints. This direction is being actively explored by several people.

Groupoids. The natural framework to present some of the ideas discussed in these pages is that of pseudogroups of transformations [24] or rather, that of their associated groupoids. Some references in this direction are [22] [15] [16].

Properties of the optimal momentum map. The standard momentum map is a fascinating object with very interesting geometrical properties. This fact suggests carrying out a similar analysis within the framework of the optimal momentum map. Such study requires some modeling of the target or momentum space of this object. Preliminary studies show that in the symplectic case this has much to do with the so called Baer groupoid [35] and with a modified version of the cylinder valued momentum maps introduced in [7]. See [29] for the details.

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