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Stability of Hamiltonian relative equilibria

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Abstract. We generalize a sufficient condition for the stability of relative equilibria in symmetric Hamiltonian systems, due to Patrick (1992 Relative equilibria in Hamiltonian systems: the dynamic interpretation of nonlinear stability on a reduced phase space *J. Geo. Phys.* **9** 111–19), to the case in which these relative equilibria have non-trivial symmetry. We also describe a block diagonalization that facilitates the use of this result in particular examples and shows the relation between the stability of the relative equilibrium and the Lyapunov stability of the associated singular reduced equilibrium.

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1. Introduction

Let $(M, \omega, G, \boldsymbol{J} : M \to \mathfrak{g}^*, h : M \to \mathbb{R})$ be a Hamiltonian dynamical system with symmetry. We assume that the Lie group G with Lie algebra $\mathfrak{g}$ acts properly and canonically on the smooth symplectic manifold (M, ω) and that the G-action admits an equivariant momentum map $\boldsymbol{J} : M \to \mathfrak{g}^*$; $\mathfrak{g}^*$ denotes the dual space of $\mathfrak{g}$. Recall that a *relative equilibrium* of the G-invariant Hamiltonian h is a point $m \in M$ such that the integral curve $m(t)$ of the Hamiltonian vector field X_h starting at m equals $\exp(t\xi) \cdot m$ for some $\xi \in \mathfrak{g}$, where $\exp : \mathfrak{g} \to G$ is the exponential map; any such ξ is called a *velocity* of the relative equilibrium. Note that if m has a non-trivial isotropy subgroup, ξ is not uniquely determined; this will become important later on.

The stability analysis of Hamiltonian relative equilibria is a very well-developed topic going back to Poincaré. An account of the modern geometric techniques for determining their stability can be found in [SLM89, SLM91, Lew92, Mar92]. The existence of symmetry gives rise to drift phenomena making non-trivial the choice of a definition of stability, given that the obvious option, orbital stability, becomes too restrictive (see the introduction of [Pat92] for an example related to the rigid body). The most natural choice is the concept of stability relative to a subgroup, explicitly introduced by Patrick [Pat92].

Definition 1.1. *Let $(M, \omega, h, G, \boldsymbol{J} : M \to \mathfrak{g}^*)$ be a Hamiltonian system with symmetry and let G' be a subgroup of G. A relative equilibrium $m \in M$ is called G'-*stable, *or* stable modulo G'*, if for any G'-invariant open neighbourhood V of the orbit $G' \cdot m$, there is an open neighbourhood $U \subseteq V$ of m, such that if F_t is the flow of the Hamiltonian vector field X_h and $u \in U$, then $F_t(u) \in V$ for all $t \geqslant 0$.*

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If G' is compact, then any open neighbourhood V of the orbit $G' \cdot m$ contains a G'-invariant open neighbourhood of $G' \cdot m$. This is a direct consequence of the slice theorem [Bre72] and the tube lemma in point set topology ([Mun75], lemma 5.8, p 169).

The same paper [Pat92, theorem 6], provides a very useful sufficient condition for the stability relative to a subgroup which involves the augmented Hamiltonian of the relative equilibrium. The hypotheses are the freeness of the G-action or, at least, the triviality of the isotropy subgroup of the relative equilibrium: $H := G_m = \{e\}$.

This paper contains three main results:

(1) We will generalize the sufficient condition for stability in [Pat92] to the case in which G is an arbitrary symmetry group and the relative equilibrium m has non-trivial symmetry (theorem 4.3). This case has been treated independently by Lerman and Singer [LS98] and, with different hypotheses, by Montaldi in [Mo97]. The approach followed in the proof consists of adapting the original proof of Patrick to the singular features of the problem.

(2) The sufficient condition referred to above involves the study of the definiteness of a bilinear form, called the stability form. We introduce a group theoretical argument to obtain a concrete block diagonalization of this form that facilitates the use of the previous result in applications (theorem 5.6).

(3) The first block of the stability form is shown to coincide with the stability form of the reduced equilibrium on the relevant symplectic stratum (theorem 5.6).

Since singular reduction and normal forms will be key ingredients in our proofs, we will dedicate two sections to briefly review the concepts and results that will be used later on. Section 4 proves the stability theorem and section 5 deals with the block diagonalization of the stability form. In the last section we illustrate these techniques and results by applying them to the stability analysis of the heavy sleeping top.

2. Singular reduction and singular relative equilibria

Singular reduction is a topic that has been developed for the last 15 years. The first studies on the structure of the symplectic reduced spaces [MW74] in the singular case are in the works of Fischer *et al* [FMM80, FMM80a], Arms *et al* [AMM81], Otto [Ot87], and Arms *et al* [ACG91]. The idea of using normal forms to describe these spaces as *stratified spaces* was first introduced by Sjamaar and Lerman [SL91], in the compact case, and by Bates and Lerman [BL97], in the case of proper actions. Other reduction schemes are presented in [AGJ90]. The proofs of the results cited below can be found in these references or in [OR98], where the point of view of point reduction, mostly used in our discussion, is explained in detail.

Let $(M, \omega, G, \boldsymbol{J} : M \to \mathfrak{g}^*, h : M \to \mathbb{R})$ be a Hamiltonian dynamical system whose symmetry is given by the Lie group G acting properly on M. The Hamiltonian $h \in C^\infty(M)$ is G-invariant and the momentum map $\boldsymbol{J}$ is assumed to be equivariant. Under these conditions, we say that M is a *Hamiltonian G-space*. If $m \in M$ is such that $\boldsymbol{J}(m) = \mu$ is a regular value of $\boldsymbol{J}$ whose coadjoint isotropy subgroup G_μ acts freely on the manifold $\boldsymbol{J}^{-1}(\mu)$, it is well known [MW74] that the space $M_\mu := \boldsymbol{J}^{-1}(\mu)/G_\mu$ is a symplectic manifold, usually called the *symplectic reduced space*, and that the dynamics induced by h reduces naturally to Hamiltonian dynamics on $\boldsymbol{J}^{-1}(\mu)/G_\mu$. When the above regularity assumptions are dropped, the *singular reduced space M_μ* is a Poisson variety in the sense of [ACG91] whose symplectic leaves are the *symplectic strata* introduced by [SL91, BL97] (see [OR98c, OR98]).

The notation that we will use is standard in the theory of group actions. Recall that the properness of the action implies that H is compact.

Proposition 2.1. *Let H and K be closed subgroups of G such that $H \subset K \subset G$. The connected components of the sets*

$$M_{(H)} = \{z \in M | G_z \text{ is conjugate to } H\}$$
$$M_{(H)}^K = \{z \in M | G_z \text{ is conjugate to } H \text{ in } K\}$$
$$M^H = \{z \in M | H \subseteq G_z\}$$
$$M_H = \{z \in M | H = G_z\} = M^H \cap M_{(H)}$$

are submanifolds of M. M_H is an open submanifold of M^H. $M_{(H)}$ is called the (H)-orbit-type *manifold. If M is symplectic, M_H and M^H are symplectic submanifolds of M. Also, for any $m \in M_H$, if $\Phi : G \times M \to M$ denotes the group action, the tangent space to M_H is given by*

$$T_m M_H = \{v_m \in T_m M | T_m \Phi_h \cdot v_m = v_m, \forall h \in H\} = T_m M^H. \tag{1}$$

Proof. See, for example, [Bre72, GS84b, BL97, Pal61]. □

If V is a representation space of H then the H-fixed point space V^H is a vector subspace of V. If, in addition, V is symplectic and H acts canonically, then V^H is a symplectic subspace of V (see [GS84b]). Thus, expression (1) can be written as $T_m M_H = T_m M^H = (T_m M)^H$, the last action being the linearized action on the tangent bundle.

Also, if H and K are closed subgroups of G such that $H \subset K \subset G$, we will denote by

$$N(H) = \{n \in G | nHn^{-1} = H\}$$
$$N_K(H) = \{n \in K | nHn^{-1} = H\} = N(H) \cap K$$

the normalizers of H in G and K respectively.

We now introduce the symplectic strata.

Theorem 2.2. *Let (M, ω) be a Hamiltonian G-space with G acting properly on M. Let $\boldsymbol{J} : M \to \mathfrak{g}^*$ be the corresponding equivariant momentum map. For $m \in M$ let $\mu := \boldsymbol{J}(m) \in \mathfrak{g}^*$ and $H := G_m$ be the isotropy subgroup of m which, by the equivariance of $\boldsymbol{J}$, is a subgroup of G_μ, the coadjoint isotropy subgroup of G at $\mu \in \mathfrak{g}^*$. Then:*

(i) The set $\boldsymbol{J}^{-1}(\mu) \cap M_H$ is a submanifold of M_H, and hence of M. Analogously, $\boldsymbol{J}^{-1}(\mu) \cap M_{(H)}^{G_\mu}$ is a submanifold of $M_{(H)}^{G_\mu}$, and therefore of M.

(ii) The set $M_\mu^{(H)} := (\boldsymbol{J}^{-1}(\mu) \cap M_{(H)}^{G_\mu})/G_\mu$ has a unique quotient differentiable structure such that the canonical projection

$$\pi_\mu^{(H)} : \boldsymbol{J}^{-1}(\mu) \cap M_{(H)}^{G_\mu} \longrightarrow M_\mu^{(H)}$$

is a surjective submersion. Endowed with this differentiable structure, $M_\mu^{(H)}$ is diffeomorphic to $(\boldsymbol{J}^{-1}(\mu) \cap M_H)/(N_{G_\mu}(H)/H)$.

(iii) There is a unique symplectic structure $\omega_\mu^{(H)}$ on $M_\mu^{(H)}$ characterized by

$$i_\mu^{(H)*}\omega = \pi_\mu^{(H)*}\omega_\mu^{(H)}$$

where $i_\mu^{(H)} : \boldsymbol{J}^{-1}(\mu) \cap M_{(H)}^{G_\mu} \hookrightarrow M$, is the natural inclusion.
The pair $(M_\mu^{(H)}, \omega_\mu^{(H)})$ is called the (H)-orbit type symplectic stratum of the reduced space M_μ.

Proof. See [SL91, BL97, OR98]. □

Lemma 2.3. *If G acts properly on M then for $G_m = H$ we have*

$$T_m(G \cdot m) \cap T_m M_H = T_m(N(H) \cdot m)$$
$$T_m(G_\mu \cdot m) \cap T_m M_H = T_m(N_{G_\mu}(H) \cdot m).$$

Proof. The first statement follows from the following chain of equalities

$$T_m(G \cdot m) \cap T_m M_H = T_m(G \cdot m) \cap (T_m M)^H = (T_m(G \cdot m))^H = T_m(G \cdot m)_H,$$

where we used (1). However,

$$(G \cdot m)_H = \{g \cdot m | G_{g\cdot m} = gHg^{-1} = H\} = \{g \cdot m | g \in N(H)\} = N(H) \cdot m.$$

So,

$$T_m(G \cdot m) \cap T_m M_H = T_m(G \cdot m)_H = T_m(N(H) \cdot m),$$

as required. The second statement follows by replacing G by G_μ. □

Proposition 2.4. *Let N be any G-invariant submanifold of M and suppose that $f \in C^\infty(N)$ is constant on each G-orbit. Then there is a smooth G-invariant extension F of f to M, that is, $F \in C^\infty(M)^G$ and $F|_N = f$.*

Proof. See [ACG91, proposition 2]. □

The spaces introduced in theorem 2.2 are suitable to reduce the dynamics induced by G-invariant Hamiltonians. For the proof see [OR98].

Theorem 2.5. *Let (M, ω) be a Hamiltonian G-space with G acting properly on M and admitting an equivariant momentum map $\boldsymbol{J} : M \to \mathfrak{g}^*$. Let $h : M \to \mathbb{R}$ be a G-invariant Hamiltonian, that is $h \circ \Phi_g = h$ for any $g \in G$. Then, using the notation of theorem 2.2:*

(i) The flow F_t of X_h leaves the connected components of $\boldsymbol{J}^{-1}(\mu) \cap M_{(H)}^{G_\mu}$ invariant and commutes with the G_μ-action, so it induces a flow F_t^μ on $M_\mu^{(H)}$ that is characterized by

$$\pi_\mu^{(H)} \circ F_t = F_t^\mu \circ \pi_\mu^{(H)}.$$

(ii) The flow F_t^μ is Hamiltonian on $M_\mu^{(H)}$, with Hamiltonian function $h_\mu^{(H)} : M_\mu^{(H)} \to \mathbb{R}$ defined by

$$h_\mu^{(H)} \circ \pi_\mu^{(H)} = h \circ i_\mu^{(H)}.$$

The vector fields X_h and $X_{h_\mu^{(H)}}$ are $\pi_\mu^{(H)}$ – related. We will call $h_\mu^{(H)}$ the reduced Hamiltonian.

(iii) Let $k : M \to \mathbb{R}$ be another G-invariant function. Then $\{h, k\}$ is also G-invariant and

$$\{h, k\}_\mu^{(H)} = \{h_\mu^{(H)}, k_\mu^{(H)}\}_{M_\mu^{(H)}}$$

where $\{\,,\,\}_{M_\mu^{(H)}}$ denotes the Poisson bracket induced by the symplectic structure in $M_\mu^{(H)}$.

For the sake of simplicity, in our next result we will require the normalizer $N(H)$ of H in G to be compact. With this assumption, we can give a very useful characterization of the symplectic strata that will be used frequently later on. The following construction is based on the fact that the Lie group $L := N(H)/H$, whose Lie algebra we denote by $\mathfrak{l}$, acts freely and properly on M_H and that $\boldsymbol{J}(M_H) \subset (\mathfrak{g}^*)^H$, where we are consistent with the notation introduced in proposition 2.1: that is, $(\mathfrak{g}^*)^H$ denotes the H-fixed vectors in $\mathfrak{g}^*$ under the coadjoint action. By compactness of $N(H)$ we can find $\mathrm{Ad}_{N(H)}$-invariant inner products on $\mathfrak{g}$ and on $\mathfrak{g}^*$ relative to which we have the orthogonal decompositions

$$\mathrm{Lie}(N(H)) = \mathfrak{h} \oplus \mathfrak{p} \qquad \text{and} \qquad \mathfrak{g}^* = \mathfrak{h}^* \oplus \mathfrak{r}^* \tag{2}$$

for some subspaces $\mathfrak{p} \subset \mathrm{Lie}(N(H)) \subset \mathfrak{g}$ and $\mathfrak{r}^* \subset \mathfrak{g}^*$. If $\lambda \in \mathfrak{l}^*$, let $\bar{\lambda} \in \mathfrak{p}$ be such that $\lambda = T_e\pi(\bar{\lambda})$, where $\pi : N(H) \to N(H)/H$ is the canonical projection onto the quotient. Then the linear map $\lambda \in \mathfrak{l} \mapsto \bar{\lambda} \in \mathfrak{p}$ is well defined, L-equivariant, has range equal to $[(\mathfrak{h}^\circ)^H]^*$ (the vector subspace of H-fixed vectors in the annihilator $\mathfrak{h}^\circ$ of $\mathfrak{h}$ in $\mathfrak{g}^*$), and is injective, so it defines an L-equivariant isomorphism $\Lambda : \mathfrak{l} \to [(\mathfrak{h}^\circ)^H]^*$ whose dual map

$$\Lambda^* : (\mathfrak{h}^\circ)^H \longrightarrow \mathfrak{l}^*$$

is hence also an L-equivariant isomorphism. Let

$$\rho : (\mathfrak{g}^*)^H \longrightarrow (\mathfrak{h}^\circ)^H$$

be the natural L-equivariant projection associated to the orthogonal decomposition (2). The L-action on M_H is canonical and has an associated equivariant momentum map $\boldsymbol{K}_L$ given by the expression

$$\boldsymbol{K}_L(z) = (\Lambda^* \circ \rho)(\boldsymbol{J}|_{M_H}(z)), \quad z \in M_H. \tag{3}$$

If $\boldsymbol{J}(m) = \mu$, we will write $\mu = \mu_{\mathfrak{h}^*} + \mu_{\mathfrak{r}^*}$ for the decomposition of μ according to the splitting (2) and will define $\lambda_\circ := \Lambda^*(\mu_{\mathfrak{r}^*}) = \boldsymbol{K}_L(m)$.

Theorem 2.6. *If $N(H)$ is compact, the symplectic stratum $(M_\mu^{(H)}, \omega_\mu^{(H)})$ is naturally symplectomorphic to the usual symplectic reduced space $(\boldsymbol{K}_L^{-1}(\lambda_\circ)/L_{\lambda_\circ}, \omega_{\lambda_\circ})$, defined by the L-action on M_H.*

Proof. See [OR98]. We will give here only the key ideas of the proof which will be used later on (for example in proposition 5.3). One begins by showing that $\boldsymbol{J}|_{M_H} : M_H \to \mathfrak{g}^*$ is a subimmersion (a constant rank map). Therefore, $(\boldsymbol{J}|_{M_H})^{-1}(\mu) = \boldsymbol{J}^{-1}(\mu) \cap M_H$ is a smooth submanifold of the symplectic manifold M_H. Next, one proves that $\lambda_0 \in \mathfrak{l}^*$ is a regular value for the L-momentum map $\boldsymbol{K}_L : M_H \to \mathfrak{l}^*$. Then one shows that $\boldsymbol{K}_L^{-1}(\lambda_\circ) = \boldsymbol{J}^{-1}(\mu) \cap M_H$ and that $L_{\lambda_\circ} = N_{G_\mu}(H)/H$ to conclude that

$$\boldsymbol{K}_L^{-1}(\lambda_\circ)/L_{\lambda_\circ} = (\boldsymbol{J}^{-1}(\mu) \cap M_H)/(N_{G_\mu}(H)/H).$$

The space on the right-hand side of this expression, as we pointed out before, is diffeomorphic to $M_\mu^{(H)}$. □

Since $\ker T_m(\boldsymbol{J}|_{M_H}) = \ker T_m\boldsymbol{J} \cap T_m M_H$ and $T_m(\boldsymbol{J}^{-1}(\mu) \cap M_H) = T_m(\boldsymbol{K}_L^{-1}(\lambda_\circ)) = \ker T_m\boldsymbol{K}_L$, we conclude from the proof above that

$$\ker T_m\boldsymbol{J} \cap T_m M_H = T_m(\boldsymbol{J}^{-1}(\mu) \cap M_H) = T_m(\boldsymbol{K}_L^{-1}(\lambda_\circ)) = \ker T_m\boldsymbol{K}_L. \tag{4}$$

Remark 2.7. If the condition on the compactness of $N(H)$ is dropped, there are still global models for the symplectic strata of the kind introduced in theorem 2.6; however, the result is more complicated since the momentum map in (3) is not equivariant and the reduction has to be carried out by correcting the coadjoint action with the cocycle given by the non-equivariance of $\boldsymbol{K}_L$.

Using these results we can give a characterization of relative equilibria, whose proof is detailed in [OR98].

Theorem 2.8. *In the hypotheses of theorems 2.2 and 2.5 the following statements are equivalent:*

(i) *The point* $m \in \boldsymbol{J}^{-1}(\mu) \cap M_{(H)}^{G_\mu}$ *with* $G_m = H$ *is a relative equilibrium.*

(ii) *The point* $[m]_\mu^{(H)} = \pi_\mu^{(H)}(m)$*, called the* singular reduced equilibrium*, is an equilibrium of the Hamiltonian system* $(M_\mu^{(H)}, \omega_\mu^{(H)}, h_\mu^{(H)})$.

(iii) *There is a unique* $\lambda \in \mathrm{Lie}(N_{G_\mu}(H)/H)$ *(the Lie algebra of* $N_{G_\mu}(H)/H$*) such that*

$$F_t(m) = \exp_L(t\lambda) \cdot m \qquad \text{for all} \quad t \in \mathbb{R}$$

where $\exp_L : \mathfrak{l} \to L$ *is the exponential map associated to the Lie group* $L := N(H)/H$ *and the dot denotes the action of* $N(H)/H$ *on* M_H.

(iv) *There is an* $\eta \in \mathrm{Lie}(N_{G_\mu}(H))$ *such that*

$$F_t(m) = \exp(t\eta) \cdot m \qquad \text{for all} \quad t \in \mathbb{R}$$

(v) *There is an* $\eta \in \mathrm{Lie}(N_{G_\mu}(H))$ *such that*

$$X_h(m) = \eta_M(m)$$

(vi) *There is an* $\eta \in \mathrm{Lie}(N_{G_\mu}(H))$ *such that the augmented Hamiltonian* $h^\eta := h - \boldsymbol{J}^\eta$ *satisfies*

$$\boldsymbol{d}h^\eta(m) = 0.$$

3. Normal forms and dynamical reconstruction

The Marle–Guillemin–Sternberg normal form (*MGS normal form*) was introduced by Marle [Mar85] and by Guillemin and Sternberg [GS84a, GS84b]. Bates and Lerman [BL97] treated in detail the proper group actions case.

The MGS normal form gives a G-invariant local model around each point of (M, ω) considered as a Hamiltonian G-space. Let $m \in M$ be as in theorem 2.2 and let $\mu := \boldsymbol{J}(m)$, $H := G_m$. The vector space $V := \ker T_m\boldsymbol{J}/T_m(G_\mu \cdot m) = T_m(G\cdot m)^\omega/T_m(G_\mu \cdot m)$ is called the *symplectic normal space* at m; it is endowed with a natural symplectic structure ω_V inherited from $\omega(m)$ and with a H-action that makes it into a Hamiltonian H-space with Ad_H^*-equivarient momentum map $J_V : V \to \mathfrak{h}^*$. By the properness of the action, H is compact, so there is an Ad_H-invariant inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$, relative to which we have the orthogonal direct sum decompositions $\mathfrak{g} = \mathfrak{g}_\mu \oplus \mathfrak{q}$ and $\mathfrak{g}_\mu = \mathfrak{h} \oplus \mathfrak{m}$ for some subspaces $\mathfrak{q} \subset \mathfrak{g}$ and $\mathfrak{m} \subset \mathfrak{g}_\mu$. The inner product also allows us to identify all these Lie algebras with their duals. In particular, we have the dual orthogonal direct sums $\mathfrak{g}^* = \mathfrak{g}_\mu^* \oplus \mathfrak{q}^*$ and $\mathfrak{g}_\mu^* = \mathfrak{h}^* \oplus \mathfrak{m}^*$ which allow an identification of $\mathfrak{g}_\mu^*$ with a subset of $\mathfrak{g}^*$ and $\mathfrak{h}^*$ and $\mathfrak{m}^*$ with subsets of $\mathfrak{g}_\mu^*$. The inclusions induced by these identifications are used in the next proposition whose proof can be found in [BL97, OR98].

Proposition 3.1. *Let* (M, ω) *be a symplectic manifold and let* G *be a Lie group acting properly on* M *in a globally Hamiltonian fashion, with* Ad^**-equivariant momentum map* $\boldsymbol{J} : M \to \mathfrak{g}^*$. *Let* $m \in M$ *and denote* $\boldsymbol{J}(m) = \mu \in \mathfrak{g}^*$. *Let* (V, ω_V) *be the symplectic normal space at* $m \in M$. *Relative to an* H*-invariant inner product on* $\mathfrak{g}$ *consider the inclusions* $\mathfrak{m}^* \subset \mathfrak{g}_\mu^* \subset \mathfrak{g}^*$. *Then there is a positive number* $r > 0$ *such that, denoting by* $\mathfrak{m}_r^*$ *the open ball of radius* r *relative to the* H*-invariant inner product on* $\mathfrak{m}^*$*, the manifold*

$$Y_r := G \times_H (\mathfrak{m}_r^* \times V)$$

can be endowed with a symplectic structure ω_{Y_r} *with respect to which the left* G*-action* $g \cdot [h, \eta, v] = [gh, \eta, v]$ *on* Y_r *is globally Hamiltonian with* Ad^**-equivariant momentum map* $\boldsymbol{J}_{Y_r} : Y_r \to \mathfrak{g}^*$ *given by*

$$\boldsymbol{J}_{Y_r}([g, \rho, v]) = \mathrm{Ad}_{g^{-1}}^* \cdot (\mu + \rho + \boldsymbol{J}_V(v)). \tag{5}$$

Remark 3.2 (The symplectic structure of Y_r). The only part in the proof of the previous proposition that will be used in what follows is the construction of the symplectic form ω_{Y_r} on Y_r which we now review.

Since H is compact and there is a well-defined free H-action on V, on $\mathfrak{m}^*$ (by the H-invariance of the inner product used to construct it), and therefore on $\mathfrak{m}^* \times V$; the twisted product $G \times_H (\mathfrak{m}^* \times V)$ is therefore a well-defined smooth manifold.

Note that, since the above-described splittings allow us to see $\mathfrak{g}_\mu^*$ as a subset of $\mathfrak{g}^*$, the manifold $Y_1 := G \times \mathfrak{g}_\mu^*$ can be considered as a submanifold of $G \times \mathfrak{g}^*$ which is diffeomorphic to the cotangent bundle T^*G (unless specified otherwise, we always use left trivializations—or body coordinates in the language of continuum mechanics—for T^*G and TG). Let ω_1 be the pull-back of the canonical symplectic form from T^*G to $G \times \mathfrak{g}^*$ and then to $G \times \mathfrak{g}_\mu^*$. Denote by ω_μ the pull-back to $G \times \mathfrak{g}_\mu^*$ of the + orbit symplectic structure $\omega_{\mathcal{O}_\mu}^+$ on the coadjoint orbit $\mathcal{O}_\mu$ through μ by the surjective submersion

$$\begin{aligned} \pi : G \times \mathfrak{g}_\mu^* &\longrightarrow \mathcal{O}_\mu \\ (g, \nu) &\longmapsto \mathrm{Ad}_{g^{-1}}^* \mu, \end{aligned}$$

and define the closed two-form Ω on Y_1 by $\Omega := \omega_1 + \omega_\mu$.

The two-form Ω is non-degenerate at each point of the form $(g, 0) \in Y_1$. Since non-degeneracy is an open condition, this will guarantee that the form Ω is non-degenerate in an open neighbourhood of the base of the trivial vector bundle $Y_1 \to G$. Thus, there is a positive number $r > 0$ such that $G \times (\mathfrak{h}_r^* \oplus \mathfrak{m}_r^*)$ is symplectic, where $\mathfrak{h}_r^*$ and $\mathfrak{m}_r^*$ are the open balls of radius r relative to the H-invariant inner products on $\mathfrak{h}^*$ and $\mathfrak{m}^*$ respectively.

Consider now the left action $\mathcal{R}$ of H on Y_1 given by

$$\mathcal{R}_h(g, \nu) = (gh^{-1}, \mathrm{Ad}_{h^{-1}}^* \nu)$$

and the orthogonal decomposition $\mathfrak{g}_\mu = \mathfrak{h} \oplus \mathfrak{m}$. Using the definition of Ω, it is straightforward to verify that this action is globally Hamiltonian (on the presymplectic manifold Y_1) with equivariant momentum map $\boldsymbol{J}_\mathcal{R} : Y_1 \to \mathfrak{h}^*$, given by

$$\boldsymbol{J}_\mathcal{R}((g, (\eta, \rho))) = -\eta, \qquad \text{for any} \quad (\eta, \rho) \in \mathfrak{h}^* \oplus \mathfrak{m}^* = \mathfrak{g}_\mu^*.$$

The H-action on V is globally Hamiltonian with momentum map $\boldsymbol{J}_V : V \to \mathfrak{h}^*$ given by the formula:

$$\langle \boldsymbol{J}_V(v), \xi \rangle = \tfrac{1}{2} \omega_V(\xi_V(v), v) \qquad \text{for any} \quad \xi \in \mathfrak{h}.$$

Putting together these two actions, we construct a product action of H on the presymplectic manifold $Y_1 \times V$ (endowed with the sum presymplectic form), which is Hamiltonian, with H-equivariant momentum map $\boldsymbol{\Phi} : Y_1 \times V \cong G \times \mathfrak{m}^* \times \mathfrak{h}^* \times V \to \mathfrak{h}^*$, given by the sum $\boldsymbol{J}_\mathcal{R} + \boldsymbol{J}_V$, that is,

$$\begin{aligned} \boldsymbol{\Phi} : G \times \mathfrak{m}^* \times \mathfrak{h}^* \times V &\longrightarrow \mathfrak{h}^* \\ (g, \rho, \eta, v) &\longmapsto \boldsymbol{J}_V(v) - \eta. \end{aligned}$$

The H-action on $Y_1 \times V$ is free and proper and $0 \in \mathfrak{h}^*$ is clearly a regular value of $\boldsymbol{\Phi}$. Therefore $\boldsymbol{\Phi}^{-1}(0)/H$ is a well-defined reduced presymplectic space which can be identified with $Y = G \times_H (\mathfrak{m}^* \times V)$ by means of the quotient diffeomorphism L, induced by the H-equivariant diffeomorphism l:

$$\begin{aligned} l : G \times \mathfrak{m}^* \times V &\longrightarrow \boldsymbol{\Phi}^{-1}(0) \subset G \times \mathfrak{m}^* \times \mathfrak{h}^* \times V \\ (g, \rho, v) &\longmapsto (g, \rho, \boldsymbol{J}_V(v), v). \end{aligned}$$

We define the presymplectic form ω_Y on Y as the pull-back by L of the reduced presymplectic form Ω_0 on $\mathbf{\Phi}^{-1}(0)/H$. Thus, we have the following commutative diagram with the lower arrow a presymplectic diffeomorphism:

$$\begin{array}{ccc} G \times \mathfrak{m}^* \times V & \stackrel{l}{\longrightarrow} & \mathbf{\Phi}^{-1}(0) \subset G \times \mathfrak{m}^* \times \mathfrak{h}^* \times V \\ \pi \downarrow & & \downarrow \pi_0 \\ (G \times_H (\mathfrak{m}^* \times V), \omega_Y) & \stackrel{L}{\longrightarrow} & (\mathbf{\Phi}^{-1}(0)/H, \Omega_0). \end{array} \tag{6}$$

It is clear that the presymplectic manifolds constructed above become symplectic submanifolds by restricting $\mathfrak{m}^*$ to $\mathfrak{m}_r^*$. This replaces Y by Y_r and $\mathbf{\Phi}^{-1}(0)/H$ by the reduction of $G \times \mathfrak{m}_r^* \times \mathfrak{h}_r^* \times V$ at zero. Thus, any statements that can be made for Y as a presymplectic manifold, can be made for Y_r as a symplectic manifold and this is why we shall work with Y in what follows.

Theorem 3.3 (MGS normal form). *Let (M, ω) be a symplectic manifold and let G be a Lie group acting properly on M in a globally Hamiltonian fashion, with associated* Ad*-*equivariant momentum map $\mathbf{J} : M \to \mathfrak{g}^*$. Let $m \in M$ and denote $\mathbf{J}(m) = \mu \in \mathfrak{g}^*$, $H := G_m$. Then the manifold*

$$Y_r := G \times_H (\mathfrak{m}_r^* \times V)$$

introduced in proposition 3.1 is a Hamiltonian G-space and there are G-invariant neighbourhoods U of m in M, U' of $[e, 0, 0]$ in Y, and an equivariant symplectomorphism $\phi : U \to U'$ satisfying $\phi(m) = [e, 0, 0]$ and $\mathbf{J}_Y \circ \phi = \mathbf{J}$.

The following proposition is based on a similar result in [BL97].

Proposition 3.4. *In the hypotheses of theorems 2.2 and 3.3, for a small enough neighbourhood Y_0 of the orbit $G \cdot [e, 0, 0]$ in the model space Y, the intersection of the set $\mathbf{J}_Y^{-1}(\mu)$ with the neighbourhood Y_0 has the form*

$$\mathbf{J}_Y^{-1}(\mu) \cap Y_0 = \{[g, \eta, v] \in Y_0 |\ g \in G_\mu,\ \eta = 0\ ,\ \mathbf{J}_V(v) = 0\}.$$

This proposition allows us to give our first local characterization of the symplectic strata.

Theorem 3.5. *The symplectic stratum $(M_\mu^{(H)}, \omega_\mu^{(H)})$ is locally symplectomorphic to $(V^H, \omega_V|_{V^H})$, where V is the symplectic normal space associated to the G-action at $m \in M$.*

Proof. See [SL91, BL97, OR98]. The proof uses proposition 3.4 in order to show that $\mathbf{J}^{-1}(\mu) \cap M_{(H)}^{G_\mu}$ can be locally represented as $G_\mu \times_H V^H \simeq (G_\mu/H) \times V^H$. This implies that $(\mathbf{J}^{-1}(\mu) \cap M_{(H)}^{G_\mu})/G_\mu \simeq V^H$. The construction of the MGS normal form guarantees that this local diffeomorphism is a symplectomorphism if V^H is endowed with the symplectic form $\omega_V|_{V^H}$. □

We will now write down, in terms of the coordinates provided by a specific normal form similar to the one in theorem 3.3, the differential equations that determine the Hamiltonian vector field X_h, where h is a G-invariant Hamiltonian. Let us fix below $m \in M$ with stabilizer $G_m = H$ and $\mu = \mathbf{J}(m)$. As we stated before, $N(H)$ is assumed to be compact.

The G-invariance of h, forces the dynamical evolution to take place on the orbit type manifolds; more specifically, if $m \in M_H$, then $F_t(m) \in M_H$ for any time t, where F_t the flow of X_h. Since M_H is a symplectic submanifold of M we can restrict ourselves to the study of the Hamiltonian system $(M_H, \omega_{|_{M_H}}, h_{|_{M_H}})$. Since, as we know, $(M_H, \omega_{|_{M_H}}, h_{|_{M_H}})$ is a Hamiltonian L-space in its own right, we can construct a MGS normal form around the orbit $L \cdot m$. It is in these coordinates that we will carry out the reconstruction of the dynamics, that is, we will write the equations that relate the dynamics in the symplectic stratum with the dynamics in the original space M.

Proposition 3.6 (MGS normal form for M_H). *Let $m \in M_H$ be an element of the Hamiltonian L-space $(M_H, \omega|_{M_H}, L, \boldsymbol{K}_L : M_H \to \mathfrak{l}^*)$, such that $G_m = H$ and $\boldsymbol{K}_L(m) = \lambda_\circ$. Assume that $N(H)$ is compact. Then*

$$Y_H := L \times \mathfrak{l}^*_{\lambda_\circ} \times V_L$$

is a presymplectic Hamiltonian L-space where L acts by $l \cdot (h, \eta, v) = (lh, \eta, v)$ with corresponding equivariant momentum map

$$\boldsymbol{K}_{L_{Y_H}}(g, \eta, v) = g \cdot (\lambda_\circ + \eta).$$

There are L-invariant neighbourhoods U of $m \in M_H$, U' of $(e, 0, 0) \in Y_H$, which is symplectic, and an equivariant symplectomorphism $\phi : U \to U'$ with $\phi(m) = (e, 0, 0)$ satisfying $\boldsymbol{K}_{L_{Y_H}} \circ \phi = \boldsymbol{K}_L$. In the previous expression, V_L is the symplectic normal space at m corresponding to the L-action, that is,

$$V_L := \frac{\ker T_m \boldsymbol{K}_L}{T_m(L_{\lambda_\circ} \cdot m)}.$$

Proof. It is a corollary to proposition 3.1 and theorem 3.3. □

This normal form, together with proposition 3.4, gives us another characterization of the symplectic stratum.

Theorem 3.7. *The symplectic stratum $(M_\mu^{(H)}, \omega_\mu^{(H)})$ is locally symplectomorphic to (V_L, ω_{V_L}), where V_L is the symplectic normal space associated to the L-action at $m \in M_H$.*

Proof. By theorem 2.6, the symplectic stratum $(M_\mu^{(H)}, \omega_\mu^{(H)})$ is naturally symplectomorphic to $(\boldsymbol{K}_L^{-1}(\lambda_\circ)/L_{\lambda_\circ}, \omega_{\lambda_\circ})$. If we use proposition 3.4 on the Hamiltonian L-space

$$Y_H := L \times \mathfrak{l}^*_{\lambda_\circ} \times V_L$$

we obtain that $\boldsymbol{K}_L^{-1}(\lambda_\circ) = L_{\lambda_\circ} \times \{0\} \times V_L \simeq L_{\lambda_\circ} \times V_L$, for a small enough neighbourhood of the orbit $L \cdot m$ and hence, $\boldsymbol{K}_L^{-1}(\lambda_\circ)/L_{\lambda_\circ} \simeq V_L$. The construction of the normal form in proposition 3.6 guarantees that this local diffeomorphism is a symplectomorphism if we endow V_L with its natural symplectic form ω_{V_L}. □

We now use the normal form to find the Hamiltonian vector field induced by $h^H := h_{|_{M_H}}$. Recall that the model for M_H is $Y_H = L \times \mathfrak{l}^*_{\lambda_\circ} \times V_L$ and that the left L-action on Y_H is only on the first factor. Given that we will study the evolution in a neighbourhood of $m = (e, 0, 0)$, we can take in the L-factor of Y_H a parametrization given by the exponential map of $\mathfrak{l}$; more specifically, we will consider an Ad_L-invariant orthogonal direct sum decomposition

$$\mathfrak{l} = \mathfrak{l}_{\lambda_0} \oplus \mathfrak{s},$$

and we will restrict ourselves to neighbourhoods of $0 \in \mathfrak{l}$ and of $e \in L$ such that the map defined by

$$\xi_{\mathfrak{l}_{\lambda_\circ}} + \xi_{\mathfrak{s}} \longmapsto \exp \xi_{\mathfrak{l}_{\lambda_\circ}} \exp \xi_{\mathfrak{s}} \longmapsto (\exp \xi_{\mathfrak{l}_{\lambda_\circ}}, \exp \xi_{\mathfrak{s}}) := (g_{\mathfrak{l}_{\lambda_\circ}}, g_{\mathfrak{s}}),$$

for $\xi_{\mathfrak{l}_{\lambda_\circ}} \in \mathfrak{l}_{\lambda_\circ}$ and $\xi_{\mathfrak{s}} \in \mathfrak{s}$, is a local diffeomorphism around zero and the identity, between the manifolds $\mathfrak{l}$, L and the product $L_{\lambda_\circ} \times (\exp \mathfrak{s})$, which we will identify in what follows. This will allow us to locally write the elements of L as pairs of the form $(g_{\mathfrak{l}_{\lambda_\circ}}, g_{\mathfrak{s}}) \in L_{\lambda_\circ} \times (\exp \mathfrak{s})$.

Analogously, with this notation, the Hamiltonian vector field X_{h^H}, associated to h^H, can be written in terms of four components:

$$X_{h^H} = (X_{\mathfrak{l}_{\lambda_\circ}}, X_{\mathfrak{s}}, X_{\mathfrak{l}^*_{\lambda_\circ}}, X_{V_L}).$$

The G-invariance of h implies that, in the normal coordinates for Y_H, h^H can be considered as a function that depends only on $\mathfrak{l}^*_{\lambda_\circ}$ and V_L. If we denote by $D_{\mathfrak{l}^*_{\lambda_\circ}} h^H$ and $D_{V_L} h^H$, the partial derivatives of h^H with respect to the factors $\mathfrak{l}^*_{\lambda_\circ}$ and V_L respectively and by ω_H the analogue for Y_H of the symplectic structure on Y introduced in remark 3.2, X_{h^H} is completely determined by the equation,

$$\boldsymbol{i}_{X_{h^H}} \omega_H = D_{\mathfrak{l}^*_{\lambda_\circ}} h^H + D_{V_L} h^H.$$

We study this equality in detail. The differential of h^H at an arbitrary point $(g_{\mathfrak{l}_{\lambda_\circ}}, g_{\mathfrak{s}}, \nu, v) \in Y_H$ in the direction $(v_{\mathfrak{l}_{\lambda_\circ}}, v_{\mathfrak{s}}, v_{\mathfrak{l}^*_{\lambda_\circ}}, v_{V_L}) \in T_{(g_{\mathfrak{l}_{\lambda_\circ}}, g_{\mathfrak{s}}, \nu, v)} Y_H$ equals hence

$$D_{\mathfrak{l}^*_{\lambda_\circ}} h^H \cdot v_{\mathfrak{l}^*_{\lambda_\circ}} + D_{V_L} h^H \cdot v_{V_L} = \omega_H (g_{\mathfrak{l}_{\lambda_\circ}}, g_{\mathfrak{s}}, \nu, v)((X_{\mathfrak{l}_{\lambda_\circ}}, X_{\mathfrak{s}}, X_{\mathfrak{l}^*_{\lambda_\circ}}, X_{V_L}), (v_{\mathfrak{l}_{\lambda_\circ}}, v_{\mathfrak{s}}, v_{\mathfrak{l}^*_{\lambda_\circ}}, v_{V_L})).$$

If we left-trivialize T^*L, we can use the left-trivialized expression for the canonical symplectic form in this space (see, for example, [AM78, proposition 4.4.1]), which yields

$$\begin{aligned} D_{\mathfrak{l}^*_{\lambda_\circ}} h^H \cdot v_{\mathfrak{l}^*_{\lambda_\circ}} + D_{V_L} h^H \cdot v_{V_L} &= \langle v_{\mathfrak{l}^*_{\lambda_\circ}}, X_{\mathfrak{l}_{\lambda_\circ}} \rangle - \langle X_{\mathfrak{l}^*_{\lambda_\circ}}, v_{\mathfrak{l}_{\lambda_\circ}} \rangle + \langle \nu, [(X_{\mathfrak{l}_{\lambda_\circ}}, X_{\mathfrak{s}}), (v_{\mathfrak{l}_{\lambda_\circ}}, v_{\mathfrak{s}})] \rangle \\ &\quad + \langle \mathrm{Ad}^*_{(g_{\mathfrak{l}_{\lambda_\circ}}, g_{\mathfrak{s}})^{-1}} \lambda_\circ, [(X_{\mathfrak{l}_{\lambda_\circ}}, X_{\mathfrak{s}}), (v_{\mathfrak{l}_{\lambda_\circ}}, v_{\mathfrak{s}})] \rangle + \boldsymbol{i}_{X_{V_L}} \omega_{V_L} \cdot v_{V_L} \\ &= \langle v_{\mathfrak{l}^*_{\lambda_\circ}}, X_{\mathfrak{l}_{\lambda_\circ}} \rangle - \langle X_{\mathfrak{l}^*_{\lambda_\circ}}, v_{\mathfrak{l}_{\lambda_\circ}} \rangle \\ &\quad + \langle \mathrm{ad}^*_{(X_{\mathfrak{l}_{\lambda_\circ}}, X_{\mathfrak{s}})} (\nu + \mathrm{Ad}^*_{(g_{\mathfrak{l}_{\lambda_\circ}}, g_{\mathfrak{s}})^{-1}} \lambda_\circ), (v_{\mathfrak{l}_{\lambda_\circ}}, v_{\mathfrak{s}}) \rangle + \boldsymbol{i}_{X_{V_L}} \omega_{V_L} \cdot v_{V_L}. \end{aligned}$$

Given that this equality is valid for any $(v_{\mathfrak{l}_{\lambda_\circ}}, v_{\mathfrak{s}}, v_{\mathfrak{l}^*_{\lambda_\circ}}, v_{V_L}) \in T_{(g_{\mathfrak{l}_{\lambda_\circ}}, g_{\mathfrak{s}}, \nu, v)} Y_H$, it implies the following four equations for $X_{h^H} = (X_{\mathfrak{l}_{\lambda_\circ}}, X_{\mathfrak{s}}, X_{\mathfrak{l}^*_{\lambda_\circ}}, X_{V_L})$:

$$\boldsymbol{i}_{X_{V_L}} \omega_{V_L} = D_{V_L} h^H \tag{7}$$

$$X_{\mathfrak{l}_{\lambda_\circ}} = D_{\mathfrak{l}^*_{\lambda_\circ}} h^H \tag{8}$$

$$\mathbb{P}_{\mathfrak{s}^*} (\mathrm{ad}^*_{(X_{\mathfrak{l}_{\lambda_\circ}}, X_{\mathfrak{s}})} (\nu + \mathrm{Ad}^*_{(g_{\mathfrak{l}_{\lambda_\circ}}, g_{\mathfrak{s}})^{-1}} \lambda_\circ)) = 0 \tag{9}$$

$$\mathbb{P}_{\mathfrak{l}^*_{\lambda_\circ}} (\mathrm{ad}^*_{(X_{\mathfrak{l}_{\lambda_\circ}}, X_{\mathfrak{s}})} (\nu + \mathrm{Ad}^*_{(g_{\mathfrak{l}_{\lambda_\circ}}, g_{\mathfrak{s}})^{-1}} \lambda_\circ)) = X_{\mathfrak{l}^*_{\lambda_\circ}} \tag{10}$$

where $\mathbb{P}_{\mathfrak{s}^*}$ and $\mathbb{P}_{\mathfrak{l}^*_{\lambda_\circ}}$ are the projections of $\mathfrak{l}^*$ on $\mathfrak{s}^*$ and $\mathfrak{l}^*_{\lambda_\circ}$ respectively, according to the dual orthogonal direct sum $\mathfrak{l}^* = \mathfrak{l}^*_{\lambda_\circ} \oplus \mathfrak{s}^*$.

Note that the first two equations determine X_{V_L} and $X_{\mathfrak{l}_{\lambda_\circ}}$ uniquely. In particular, by theorem 3.7, the first equation is determined by the dynamics in the symplectic stratum. Therefore, the reconstruction consists of knowing X_{V_L}, finding $X_{\mathfrak{l}_{\lambda_\circ}}$, $X_{\mathfrak{s}}$, and $X_{\mathfrak{l}^*_{\lambda_\circ}}$, which are determined by the last three equations.

4. Singular relative equilibria and G_μ-stability

With the tools introduced in the previous sections we start the study of the stability of relative equilibria. The first new concept needed is the *orthogonal velocity* of a singular relative equilibrium. If $m \in M$ is a relative equilibrium of the Hamiltonian system with symmetry $(M, \omega, h, G, \boldsymbol{J} : M \to \mathfrak{g}^*)$, where G acts properly on M, $H := G_m$, and $\boldsymbol{J}(m) = \mu$, we know (see theorem 2.8 (iii)) that there is a unique $\lambda \in \mathrm{Lie}(N_{G_\mu}(H)/H)$ such that

$$F_t(m) = \exp_L t\lambda \cdot m$$

where F_t is the Hamiltonian flow of X_h and the dot denotes the action of $N_{G_\mu}(H)/H$ on M_H. The properness of the G-action allows us to choose an Ad_H-invariant inner product in $\mathfrak{n}_\mu := \mathrm{Lie}(N_{G_\mu}(H))$ and hence we have an orthogonal direct sum decomposition

$$\mathfrak{n}_\mu = \mathfrak{h} \oplus \mathfrak{p}_\mu \tag{11}$$

where $\mathfrak{p}_\mu$ is the orthocomplement of $\mathfrak{h}$ in $\mathfrak{n}_\mu$ relative to the inner product on $\mathfrak{n}_\mu$. From here it follows that

$$\mathrm{Lie}(N_{G_\mu}(H)/H) \simeq \mathfrak{n}_\mu/\mathfrak{h} \simeq \mathfrak{p}_\mu. \tag{12}$$

Let $\xi \in \mathfrak{p}_\mu \subset \mathfrak{n}_\mu$ be the unique image of $\lambda \in \mathrm{Lie}(N_{G_\mu}(H)/H)$, under the isomorphism in (12). Since the projection of $N_{G_\mu}(H)$ onto $N_{G_\mu}(H)/H$ is a group homomorphism, we can write

$$F_t(m) = \exp_L t\lambda \cdot m = \exp t\xi \cdot m.$$

Definition 4.1. *The unique element $\xi \in \mathfrak{p}_\mu$ just defined is called the orthogonal velocity of the relative equilibrium $m \in M$, relative to the splitting (11).*

Remark 4.2. Note that the orthogonal velocity depends on the splitting (11) and is unique only if this splitting is specified. Therefore it is not uniquely determined by the relative equilibrium. In applications, probing the stability of the system with all its possible orthogonal velocities, that is, considering all possible splittings (11), is the way to obtain optimal stability conditions. See section 6 for an illustration of this comment.

We can state now our first main result.

Theorem 4.3. *Let $(M, \{\cdot,\cdot\}, h)$ be a Poisson system with a symmetry given by the Lie group G acting properly on M in a globally Hamiltonian fashion, with associated equivariant momentum map $\boldsymbol{J} : M \to \mathfrak{g}^*$. Assume that the Hamiltonian $h \in C^\infty(M)$ is G-invariant. Let $m \in M$ be a relative equilibrium such that $\boldsymbol{J}(m) = \mu \in \mathfrak{g}^*$, $\mathfrak{g}^*$ admits an $\mathrm{Ad}^*_{G_\mu}$-invariant inner product, $H := G_m$, and $\xi \in \mathrm{Lie}(N_{G_\mu}(H))$ is its orthogonal velocity, relative to a given Ad_H-invariant splitting. If the bilinear form*

$$\boldsymbol{d}^2(h - \boldsymbol{J}^\xi)(m)|_{W\times W}$$

is definite for some (and hence for any) subspace W such that

$$\ker T_m\boldsymbol{J} = W \oplus T_m(G_\mu \cdot m),$$

then m is a G_μ-stable relative equilibrium. If $\dim W = 0$*, then m is always a G_μ-stable relative equilibrium. The bilinear form $\boldsymbol{d}^2(h - \boldsymbol{J}^\xi)(m)|_{W\times W}$, will be called the* stability form *of the relative equilibrium m.*

Proof. In what follows we reproduce Patrick's proof [Pat92] with a modified velocity map (see below) that makes it suitable for the singular case. We first suppose that $W \neq \{0\}$. It is easy to show that the result does not depend on the choice of point in the trajectory of the relative equilibrium m. The choice of W is also irrelevant since $\boldsymbol{d}^2(h - \boldsymbol{J}^\xi)(m)(v, w) = 0$, whenever $v \in T_m(G_\mu \cdot m)$, and $w \in \ker T_m\boldsymbol{J}$. Indeed, if we take $v = \eta_M(m)$ with $\eta \in \mathfrak{g}_\mu$, then

$$\boldsymbol{d}^2(\mathrm{h} - \boldsymbol{J}^\xi)(m)(v, w) = w[\mathrm{X}_{\boldsymbol{J}^\eta}[\mathrm{h} - \boldsymbol{J}^\xi]] = w[\{\mathrm{h}, \boldsymbol{J}^\eta\} - \boldsymbol{J}^{[\xi,\eta]}] = w[\boldsymbol{J}^{[\xi,\eta]}] = 0,$$

where we used the G-invariance of h, and that $w \in \ker T_m\boldsymbol{J}$.

Let S be a slice at the point m associated to the Hamiltonian action of G_μ on M. Let now $T := G_\mu \cdot S$, be a tube around the orbit $G_\mu \cdot m$. By definition,

$$T_mM = T_mS \oplus T_m(G_\mu \cdot m). \tag{13}$$

Let now

$$Z := T_m S \cap \ker T_m \boldsymbol{J}. \tag{14}$$

Since $T_m(G_\mu \cdot m) \subset \ker T_m \boldsymbol{J}$, by (13) and (14), we have that

$$\ker T_m \boldsymbol{J} = Z \oplus T_m(G_\mu \cdot m);$$

hence Z satisfies the requirements of W in the statement of the theorem.

We now introduce a singular *Patrick velocity map*. We start with the following lemma.

Lemma 4.4. *Fix a splitting (11) and let $\xi \in \mathfrak{p}_\mu$ be the corresponding orthogonal velocity of the relative equilibrium $m \in M$ whose symmetry group is $H := G_m$. Then $\mathrm{Ad}_h \xi = \xi$ for any $h \in H$.*

Proof. By definition

$$h \cdot \xi := \mathrm{Ad}_h \xi = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} h \exp t\xi h^{-1} = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \exp t\xi h'(t) h^{-1} \tag{15}$$

where $h'(t)$ is some element in H such that $h \exp t\xi = \exp t\xi h'(t)$ and whose existence is guaranteed by the fact that $\exp t\xi \in N_{G_\mu}(H)$. By construction, $h'(t)h^{-1}$ is a curve in H through the identity and hence there is a $\sigma \in \mathfrak{h}$ such that

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} h'(t)h^{-1} = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \exp t\sigma = \sigma.$$

Using the Leibniz rule in (15) we get

$$h \cdot \xi = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \exp t\xi h'(t) h^{-1} = \xi + \sigma.$$

Since $\xi \in \mathfrak{p}_\mu$, the Ad_H-invariance of the splitting (11) implies that $h \cdot \xi \in \mathfrak{p}_\mu$. The above identity and the splitting (11) forces $\sigma \in \mathfrak{h} \cap \mathfrak{p}_\mu = \{0\}$ which then implies $h \cdot \xi = \xi$. This proves lemma 4.4. □

Returning to the proof of the theorem, let r be the G_μ-equivariant retraction associated to the slice S [Pal61]

$$\begin{aligned} r : G_\mu \cdot S &\longrightarrow G_\mu \cdot m \\ g \cdot z &\longmapsto g \cdot m. \end{aligned}$$

We define

$$\begin{aligned} \tilde{\Psi} : G_\mu \cdot m &\longrightarrow G_\mu \cdot \xi \\ g \cdot m &\longmapsto \mathrm{Ad}_g \xi \end{aligned}$$

with ξ the orthogonal velocity of the relative equilibrium. The previous lemma guarantees that $\tilde{\Psi}$ is well-defined: if $g \cdot m = g' \cdot m$ then $g^{-1}g' \in H$ and therefore $g^{-1}g' \cdot \xi = \xi$ and so $g' \cdot \xi = g \cdot \xi$. We define the singular *Patrick velocity map* as $\Psi := \tilde{\Psi} \circ r : g \cdot z \in G_\mu \cdot S \mapsto \mathrm{Ad}_g \xi \in G_\mu \cdot \xi$. Note that $\Psi(m) = \tilde{\Psi}(m) = \xi$ and that for any $g \in G_\mu$ and any $z = g' \cdot z' \in G_\mu \cdot S$,

$$\Psi(g \cdot z) = \Psi(gg' \cdot z') = \mathrm{Ad}_{gg'} \xi = \mathrm{Ad}_g(\mathrm{Ad}_{g'} \xi) = \mathrm{Ad}_g \Psi(g' \cdot z') = \mathrm{Ad}_g \Psi(z).$$

Also, $\mathrm{Im}\, \Psi = G_\mu \cdot \xi$ and $\langle \mu, \Psi(z) \rangle = \langle \mu, \xi \rangle$, for any $z \in G_\mu \cdot S$. Note that the properties of Ψ just mentioned are the only features of the velocity map needed in Patrick's proof [Pat92]. For the sake of completeness we reproduce here the original argument.

Let f_1 and f_2 be the functions defined by

$$f_1 = (h - h(m)) + (\langle \boldsymbol{J}, \Psi\rangle - \langle \mu, \xi\rangle),$$
$$f_2 = \|\boldsymbol{J} - \mu\|^2,$$

where in f_2, the modulus is taken using the norm associated to some $\mathrm{Ad}^*_{G_\mu}$-invariant inner product in $\mathfrak{g}^*$ (always available by hypothesis), that makes f_2 a G_μ-invariant conserved quantity. Remark that f_1 is G_μ-invariant but, in general it is not conserved. Notice also that on S, $h - \boldsymbol{J}^\xi$ and $f_1|_S$ differ by a constant, which implies that $\boldsymbol{d}(f_1|_S)(m) = 0$ and $\boldsymbol{d}^2(f_1|_S)(m)$ is well-defined. Moreover,

$$\boldsymbol{d}^2(f_1|_S)(m)|_{Z\times Z} = \boldsymbol{d}^2(h - \boldsymbol{J}^\xi)(m)|_{Z\times Z}.$$

Since Z satisfies the requirements of W, the hypotheses of the theorem guarantees that $\boldsymbol{d}^2(f_1|_S)(m)|_{Z\times Z}$ is definite.

We now prove that Z is the kernel of $\boldsymbol{d}^2(f_2|_S)(m)$. It is easy to see that if $v_1, v_2 \in T_mS$ then

$$\boldsymbol{d}^2(f_2|_S)(m)(v_1, v_2) = 2\|T_m\boldsymbol{J}\cdot v_1\|\|T_m\boldsymbol{J}\cdot v_2\|.$$

Then, $v_1 \in \ker \boldsymbol{d}^2(f_2|_S)(m)$ iff for any $v_2 \in T_mS$, we have that $\|T_m\boldsymbol{J}\cdot v_1\|\|T_m\boldsymbol{J}\cdot v_2\| = 0$. In particular, for $v_1 = v_2$, this identity implies that $\|T_m\boldsymbol{J}\cdot v_1\| = 0$ and hence $v_1 \in \ker T_m\boldsymbol{J} \cap T_mS = Z$. Conversely, if $v_1 \in Z = T_mS \cap \ker T_m\boldsymbol{J}$ the above relation is satisfied trivially for all $v_1 \in T_mS$. Therefore,

$$Z = \ker \boldsymbol{d}^2(f_2|_S)(m).$$

The following lemma from [Pat92, lemma 3], is needed in what follows.

Lemma 4.5 (Patrick). *Let A and B be bilinear forms on a finite-dimensional vector space. Suppose that A is positive semidefinite and that B is positive definite on* $\ker A$. *Then there exists $r > 0$ such that $A + \epsilon B$ is positive definite for all $\epsilon \in (0, r)$.*

This lemma guarantees the existence of a positive constant $a > 0$ for which

$$f := af_1 + f_2$$

such that $\boldsymbol{d}^2(f|_S)(m)$ is positive definite and $f \geqslant 0$ in a given neighbourhood of the point m. Note that f is G_μ-invariant but, in general, it is not a constant of the motion since $\langle \boldsymbol{J}, \Psi\rangle$ is not conserved. In fact, for any $z \in S$ such that $F_t(z) \in G_\mu \cdot S$, we have

$$\begin{aligned}
\frac{1}{a}(f(F_t(z)) - f(z)) &= \langle \boldsymbol{J}(F_t(z)), \Psi(F_t(z))\rangle - \langle \boldsymbol{J}(z), \Psi(z)\rangle \\
&= \langle \boldsymbol{J}(z), \Psi(F_t(z)) - \Psi(z)\rangle \\
&= \langle \boldsymbol{J}(z) - \mu + \mu, \Psi(F_t(z)) - \xi\rangle \\
&= \langle \boldsymbol{J}(z) - \mu, \Psi(F_t(z)) - \xi\rangle + \langle \mu, \Psi(F_t(z))\rangle - \langle \mu, \xi\rangle \\
&= \langle \boldsymbol{J}(z) - \mu, \Psi(F_t(z)) - \xi\rangle,
\end{aligned}$$

where we used Noether's Theorem, $\Psi(z) = \xi$ because $z \in S$, and $\langle \mu, \Psi(z)\rangle = \langle \mu, \xi\rangle$, for any $z \in G_\mu \cdot S$. Hence, for any $z \in S$ such that $F_t(z) \in G_\mu \cdot S$,

$$\begin{aligned}
0 \leqslant f(F_t(z)) \leqslant f(z) + a|\langle \boldsymbol{J}(z) - \mu, \Psi(F_t(z)) - \xi\rangle| \leqslant f(z) + a\|\boldsymbol{J}(z) \\
-\mu\|(\|\Psi(F_t(z))\| + \|\xi\|) = f(z) + 2a\|\xi\|\|\boldsymbol{J}(z) - \mu\|, \qquad (16)
\end{aligned}$$

where we used that $\mathrm{Im}\,\Psi = G_\mu \cdot \xi$, and the G_μ-invariance of the norm $\|\cdot\|$.

With these tools, we prove the G_μ-stability of m. Let V be a G_μ-invariant open neighbourhood of $G_\mu \cdot m$. Since $f(m) = 0$, by the positive definiteness of $\boldsymbol{d}^2(f|_S)(m)$ and the Morse lemma, there is an $\epsilon > 0$ such that

$$f^{-1}[0,\epsilon) \cap S \subset V \qquad \text{and} \qquad \overline{f^{-1}[0,\epsilon) \cap S} \subset S, \tag{17}$$

where $f^{-1}[0,\epsilon)$ is an open neighbourhood of m in S. The continuity of f and $\boldsymbol{J}$, as well as the compactness of the isotropy subgroup H, imply the existence of an open H-invariant neighbourhood S' of m in S such that $S' \subset f^{-1}[0,\epsilon) \cap S$, and that for any $z \in S'$, $f(z) < \epsilon/2$ and $\|\boldsymbol{J}(z) - \mu\| < \epsilon/4a\|\xi\|$. We shall prove that

$$F_t(S') \subset f^{-1}[0,\epsilon) \cap G_\mu \cdot S, \qquad \text{for all positive } t. \tag{18}$$

Given this inclusion $U := \cup_{t \geqslant 0} F_t(G_\mu \cdot S') \subset f^{-1}[0,\epsilon) \cap G_\mu \cdot S \subset V$ is the neighbourhood that we need to conclude stability, in other words, $F_t(U) \subset V$ for all $t \geqslant 0$.

We will show the inclusion (18) by contradiction. Suppose that (18) is false for some positive t, which implies the existence of a $z_0 \in S'$ such that

$$t_0 := \sup\{t \geqslant 0 | F_s(z_0) \in f^{-1}[0,\epsilon) \cap G_\mu \cdot S, \forall s \in [0,t)\} < \infty.$$

The point $p_0 := F_{t_0}(z_0) \notin f^{-1}[0,\epsilon) \cap G_\mu \cdot S$ by the openness of $f^{-1}[0,\epsilon) \cap G_\mu \cdot S$ and the definition of t_0. However, by construction $p_0 \in \overline{f^{-1}[0,\epsilon) \cap G_\mu \cdot S}$. Thus, there are sequences $\{z_i\} \subset S$ and $\{g_i\} \subset G_\mu$ such that $g_i \cdot z_i \to p_0$. Since $\{g_i \cdot z_i\} \subset f^{-1}[0,\epsilon) \cap G_\mu \cdot S$ and f is G_μ-invariant, the sequence $\{z_i\}$ actually belongs to $f^{-1}[0,\epsilon) \cap S$. Since S is relatively compact, we may assume that $z_i \to z \in \overline{f^{-1}[0,\epsilon) \cap S} \subset S$, by (17). At the same time, the properness of the G_μ action gives a subsequence of $\{g_i\}$ converging to some $g \in G_\mu$. Therefore, $F_{t_0}(z_0) = p_0 = g \cdot z \in G_\mu \cdot S$ so that we can apply inequality (16) to get that

$$f(p_0) = f(F_{t_0}(z_0)) \leqslant f(z_0) + 2a\|\xi\|\|\boldsymbol{J}(z_0) - \mu\| < \epsilon,$$

since $z_0 \in S'$. We conclude that $p_0 \in f^{-1}[0,\epsilon) \cap G_\mu \cdot S$, which is a contradiction.

If $W = \{0\}$, then $\boldsymbol{d}^2(f_1|_S)(m)$ is definite. The theorem follows, taking f_1 as f in the previous proof. □

Remark 4.6. The hypothesis on the existence of an $\mathrm{Ad}^*_{G_\mu}$-invariant inner product on $\mathfrak{g}^*$ cannot be dropped. If the coadjoint isotropy subgroup G_μ is compact, it is always satisfied. However, the following counterexample, that we owe to the referee, illustrates the necessity of this hypothesis. Let $G = SL(2,\mathbb{R})$, $M = G \times \mathfrak{g}^*$ with the canonical symplectic form of the left trivialization of T^*G, $\xi = \mathrm{diag}(1,-1)$, the Hamiltonian $h(g,\mu) = \langle \mu, \xi \rangle$, and the left action $g \cdot (h,\mu) = (gh,\mu)$. Note that this action is proper by [tD87, proposition 5.3] and that this Hamiltonian system exhibits a relative equilibrium at the point $(e,0)$ with velocity ξ. It can be checked that in this particular situation $\dim W = 0$ which implies, if theorem 4.3 was valid, that $(e,0)$ is stable. This is not the case. Indeed, if we study the dynamics around the reduced equilibrium, the origin in the Lie–Poisson space $M/G = \mathfrak{sl}(2,\mathbb{R})^*$, we conclude that it is unstable, which prevents $(e,0)$ from being G-stable. This is seen by observing that the restriction of the dynamics to the cone through the origin (a union of three symplectic leaves) has a saddle at that point.

Remark 4.7. Theorem 4.3 has been proven independently in [LS98] using a different approach based on normal forms. Other results dealing with singular relative equilibria can be found in [Mo97].

5. Block diagonalization of the stability form

In theorem 4.3 we proved that one way to insure the stability modulo G_μ of a relative equilibrium was showing the definiteness of the Hessian of one of its associated augmented Hamiltonians in a complement to $T_m(G_\mu \cdot m)$ in $\ker T_m\boldsymbol{J}$. This is what we called the *stability form*. In applications, the size of this form may be large; therefore, it will be convenient to express it in block diagonal form, if possible, to make the definiteness analysis simpler. In what follows, we give a method to achieve such a block diagonalization.

A tool that will be of great importance below is the *isotypic decomposition* of a real representation space of a compact group. For a proof see, for instance, [GSS85] (theorems 2.5 and 3.5 of chapter 12)†.

Theorem 5.1. *Let W be a real representation space of the compact Lie group H. Then:*

(i) Up to H-isomorphisms, there are a finite number of distinct (that is, not H-isomorphic) H-irreducible subspaces of W. Call these $U_1, \ldots, U_t$.

(ii) Define W_k to be the sum of all H-irreducible subspaces V of W such that V is H-isomorphic to U_k. Then

$$W = W_1 \oplus \cdots \oplus W_t.$$

*We say that the above direct sum decomposition is the H-*isotypic decomposition *of W and that W_k is the* isotypic component *of W of type U_k. By construction, this decomposition is unique.*

(iii) Let $A : W \to W$ be a H-equivariant linear mapping. Then $A(W_k) \subset W_k$ for $k = 1, \ldots, t$.

The following corollary, which will be applied later to the stability form, is a trivial consequence of part (iii) in the previous theorem.

Corollary 5.2. *In the hypothesis of the previous theorem, let $B : W \times W \to \mathbb{R}$ be a H-invariant bilinear form. Then, the matricial expression of B with respect to the isotypic decomposition of W block diagonalizes, each block corresponding to the restriction of B to an isotypic component.*

We now turn to the study of the stability form. Our starting point will be a splitting due to Arms *et al* [AFM75] (*AFM splitting*) generalizing the well-known Moncrief decomposition which will be recalled below. In all that follows we fix $m \in M$ and denote $H := G_m$. Since H is compact, we can always find an H-invariant metric $\langle \cdot, \cdot \rangle$ on M. We define $T_m^*\boldsymbol{J} : \mathfrak{g} \to T_mM$ to be the dual map of $T_m\boldsymbol{J} : T_mM \to \mathfrak{g}^*$ followed by the identification of T_m^*M with T_mM by the metric, that is,

$$\langle T_m^*\boldsymbol{J}(\xi), v_m \rangle = T_m\boldsymbol{J}(v_m) \cdot \xi \qquad \text{for all} \quad \xi \in \mathfrak{g}, \quad v_m \in T_mM. \tag{19}$$

Note that this definition implies that

$$\ker T_m^*\boldsymbol{J} = (\text{range } T_m\boldsymbol{J})^\circ,$$

where

$$(\text{range } T_m\boldsymbol{J})^\circ = \{\xi \in \mathfrak{g} | T_m\boldsymbol{J}(v_m) \cdot \xi = 0 \text{ for all } v_m \in T_mM\}$$

is the annihilator in $\mathfrak{g}$ of range $T_m\boldsymbol{J} \subset \mathfrak{g}^*$. In particular, $\dim(\text{range } T_m^*\boldsymbol{J}) = \dim(M) - \dim(\ker T_m^*\boldsymbol{J}) = \dim(M) - \dim((\text{range } T_m\boldsymbol{J})^\circ) = \dim(\text{range } T_m\boldsymbol{J})$. In addition, a simple verification shows that range $T_m^*\boldsymbol{J} \subseteq (\ker T_m\boldsymbol{J})^\perp$, where $(\ker T_m\boldsymbol{J})^\perp$ is the orthogonal

† We thank M Roberts for suggesting to us the use of this approach.

complement of $\ker T_m\boldsymbol{J}$ in T_mM relative to the H-invariant metric $\langle\cdot,\cdot\rangle$. These two facts immediately imply

$$\text{range } T_m^*\boldsymbol{J} = (\ker T_m\boldsymbol{J})^\perp \tag{20}$$

and hence

$$T_mM = \ker T_m\boldsymbol{J} \oplus \text{range } T_m^*\boldsymbol{J}. \tag{21}$$

We also have the linear map

$$\begin{aligned}\alpha_m : \mathfrak{g}_\mu &\longrightarrow T_mM\\ \xi &\longmapsto \xi_M(m)\end{aligned}$$

and proceeding as before we conclude

$$T_mM = \text{range } \alpha_m \oplus \ker \alpha_m^* \tag{22}$$

where $\alpha_m^* : T_mM \to \mathfrak{g}_\mu^*$ is defined by $\alpha_m^*(v_m)\cdot\eta = \langle v_m, \alpha_m(\eta)\rangle$ for all $\eta \in \mathfrak{g}_\mu$, $v_m \in T_mM$. The infinitesimal equivariance of $\boldsymbol{J}$ states that $T_m\boldsymbol{J}\cdot\xi_M(m) = -\text{ad}_\xi^*\boldsymbol{J}(m)$, which implies range $\alpha_m \subset \ker T_m\boldsymbol{J}$. Thus (22) implies the orthogonal direct sum decomposition

$$\ker T_m\boldsymbol{J} = \text{range } \alpha_m \oplus (\ker T_m\boldsymbol{J} \cap \ker \alpha_m^*) \tag{23}$$

and hence the finer orthogonal decomposition of T_mM [AFM75, théorème 2.3],

$$\begin{aligned}T_mM &= \text{range } \alpha_m \oplus (\ker T_m\boldsymbol{J} \cap \ker\alpha_m^*) \oplus \text{range } T_m^*\boldsymbol{J}\\ &= T_m(G_\mu\cdot m) \oplus (\ker T_m\boldsymbol{J} \cap T_m(G_\mu\cdot m)^\perp) \oplus \text{range } T_m^*\boldsymbol{J}\\ &\simeq T_m(G_\mu\cdot m) \oplus \ker T_m\boldsymbol{J}/T_m(G_\mu\cdot m) \oplus T_m(G\cdot m),\end{aligned} \tag{24}$$

where the last line should be understood as an external direct sum. We define

$$W := \ker T_m\boldsymbol{J} \cap \ker\alpha_m^* = \ker T_m\boldsymbol{J} \cap T_m(G_\mu\cdot m)^\perp.$$

By the Reduction Lemma [AM78, lemma 4.3.2], we have

$$\ker T_m\boldsymbol{J} = T_m(G\cdot m)^\omega \qquad \text{and} \qquad T_m(G\cdot m)\cap\ker T_m\boldsymbol{J} = T_m(G_\mu\cdot m) \tag{25}$$

which implies that

$$\phi_m((\ker T_m\boldsymbol{J})^\perp) = \omega_m(T_m(G\cdot m)) \qquad \text{and} \qquad \phi_m(T_m(G\cdot m)^\perp) = \omega_m(\ker T_m\boldsymbol{J}), \tag{26}$$

where $\phi_m : T_mM \to T_m^*M$ and $\omega_m : T_mM \to T_m^*M$ are the isomorphisms associated to the metric and the symplectic structure, respectively. Indeed, by (20), any element of $\phi_m(\ker(T_m\boldsymbol{J})^\perp)$ is of the form $\langle T_m^*\boldsymbol{J}\cdot\xi,\cdot\rangle$, for some $\xi \in \mathfrak{g}$. By (19) and the definition of the momentum map, for any $v_m \in T_mM$, we have $\langle T_m^*\boldsymbol{J}\cdot\xi, v_m\rangle = T_m\boldsymbol{J}(v_m)\cdot\xi = \omega(m)(\xi_M(m), v_m)$, which is equivalent to the first identity above. The second is proved in a similar way.

Proposition 5.3. *The subspace $W = \ker T_m\boldsymbol{J}\cap\ker\alpha_m^* = \ker T_m\boldsymbol{J}\cap T_m(G_\mu\cdot m)^\perp$, constructed using the H-invariant metric $\langle\cdot,\cdot\rangle$, has the following properties:*

- *(i) $(W, \omega(m)|_W)$ is a symplectic subspace of $(T_mM, \omega(m))$.*
- *(ii) W is H-invariant.*
- *(iii) The symplectic vector subspace $(W^H, \omega(m)|_{W^H})$ is naturally symplectomorphic to the tangent space $(T_{[m]_\mu^{(H)}}M_\mu^{(H)}, \omega_\mu^{(H)}([m]_\mu^{(H)}))$ of the symplectic stratum $(M_\mu^{(H)}, \omega_\mu^{(H)})$.*

Proof.

(i) Let $v \in W$ such that $\omega(m)(v, w) = 0$ for all $w \in W$. If $u = \eta_M(m) \in$ range α_m for some $\eta \in \mathfrak{g}_\mu$, then $\omega(m)(v, u) = 0$, since $v \in \ker T_m\boldsymbol{J} = T_m(G \cdot m)^\omega$, by (25). This implies that $v \in (\text{range } \alpha_m)^\omega \cap W^\omega = (\text{range } \alpha_m \oplus W)^\omega = (\ker T_m\boldsymbol{J})^\omega = T_m(G \cdot m)$, by (23) and (25). Since $v \in W \subset \ker T_m\boldsymbol{J}$, we therefore have $v \in \ker T_m\boldsymbol{J} \cap T_m(G \cdot m) = T_m(G_\mu \cdot m) =$ range α_m, again by (25). But range $\alpha_m \cap W = \{0\}$ and hence $v = 0$.

(ii) Let $v \in W$ and $h \in H \subset G_\mu$ be arbitrary. By H-invariance of the metric $\langle\cdot,\cdot\rangle$, for any $\eta_M(m) \in$ range α_m, that is, $\eta \in \mathfrak{g}_\mu$, we have

$$\langle h \cdot v, \eta_M(m)\rangle = \langle v, h^{-1} \cdot \eta_M(m)\rangle = \langle v, (\mathrm{Ad}_{h^{-1}}\eta)_M(m)\rangle = 0 \tag{27}$$

since $v \in W \subset (T_m(G_\mu \cdot m))^\perp$, $h \cdot m = m$, and $\mathrm{Ad}_{h^{-1}}\eta \in \mathfrak{g}_\mu$. Thus, $h \cdot v \in (\text{range } \alpha_m)^\perp$. Analogously, if $w \in (\ker T_m\boldsymbol{J})^\perp$, by (26), there is a $\lambda \in \mathfrak{g}$ for which

$$\begin{aligned}\langle h \cdot v, w\rangle &= \omega(m)(h \cdot v, \lambda_M(m)) = \omega(m)(v, (\mathrm{Ad}_{h^{-1}}\lambda)_M(m)) \\ &= -T_m\boldsymbol{J}(v) \cdot \mathrm{Ad}_{h^{-1}}\lambda = 0,\end{aligned} \tag{28}$$

by H-invariance of ω, the definition of the momentum map, and since $v \in W \subset \ker T_m\boldsymbol{J}$. Thus, $h \cdot v \in \ker T_m\boldsymbol{J}$ and we conclude that $h \cdot v \in \ker T_m\boldsymbol{J} \cap (\text{range } \alpha_m)^\perp = W$.

(iii) For simplicity, we will give the proof under the additional assumption that the normalizer $N(H)$ is compact in which case, by theorem 2.6, $(M_\mu^{(H)}, \omega_\mu^{(H)})$ is symplectomorphic to $(\boldsymbol{K}_L^{-1}(\lambda_\circ)/L_{\lambda_\circ}, \omega_{\lambda_\circ})$. This assumption is not essential since in the general case there are similar global models for the symplectic strata based on nonequivariant symplectic reduction [OR98]. Using the notation introduced in the proof of that result, the surjective submersion $\pi_{\lambda_\circ} : \boldsymbol{K}_L^{-1}(\lambda_\circ) \to \boldsymbol{K}_L^{-1}(\lambda_\circ)/L_{\lambda_\circ}$ allows us to identify $T_{[m]_\mu^{(H)}}M_\mu^{(H)}$ with $T_m\boldsymbol{K}_L^{-1}(\lambda_\circ)/T_m(L_{\lambda_\circ} \cdot m)$. By the definition of W^H and (4), we have

$$\begin{aligned}W^H &= \ker T_m\boldsymbol{J} \cap T_mM_H \cap T_m(G_\mu \cdot m)^\perp \\ &= T_m\boldsymbol{K}_L^{-1}(\lambda_\circ) \cap T_m(G_\mu \cdot m)^\perp \subset T_m\boldsymbol{K}_L^{-1}(\lambda_\circ).\end{aligned}$$

We will show that the linear map

$$\begin{aligned}\Delta : W^H &\longrightarrow T_m\boldsymbol{K}_L^{-1}(\lambda_\circ)/T_m(L_{\lambda_\circ} \cdot m) \\ w &\longmapsto w + T_m(L_{\lambda_\circ} \cdot m)\end{aligned}$$

is an isomorphism.

If $w + T_m(L_{\lambda_\circ} \cdot m) = T_m(L_{\lambda_\circ} \cdot m)$, then $w = \lambda_M(m)$ for some $\lambda \in \mathfrak{l}_{\lambda_0}$. Since $L_{\lambda_\circ} = N_{G_\mu}(H)/H$, there is a $\nu \in \mathrm{Lie}(N_{G_\mu}(H)) \subset \mathfrak{g}_\mu$ which projects to λ. Therefore, by the definition of the action of L_{λ_0}, we have $\nu_M(m) = \lambda_M(m) = w \in W$. Since $W \cap T_m(G_\mu \cdot m) = \{0\}$, it follows that $\nu_M(m) = 0$ and hence $w = 0$, proving that $\ker\Delta = \{0\}$ and hence injectivity of Δ. We now show that Δ is surjective. Let $w + T_m(L_{\lambda_\circ} \cdot m) \in T_m\boldsymbol{K}_L^{-1}(\lambda_\circ)/T_m(L_{\lambda_\circ} \cdot m)$ with $w \in T_m\boldsymbol{K}_L^{-1}(\lambda_\circ) \subset \ker T_m\boldsymbol{J}$. By the AFM decomposition (23), w can be uniquely expressed as an orthogonal sum

$$w = \eta_M(m) + w' \qquad \text{with} \quad \eta \in \mathfrak{g}_\mu \qquad \text{and} \quad w' \in W.$$

By (4), $T_m\boldsymbol{K}_L^{-1}(\lambda_\circ) = \ker T_m\boldsymbol{J} \cap T_mM_H \subset (T_mM)^H$ and hence w is also a H-fixed vector, that is,

$$h \cdot w = w \qquad \text{for all} \quad h \in H.$$

We therefore conclude that

$$h \cdot \eta_M(m) + h \cdot w' = \eta_M(m) + w' \qquad \text{for all} \quad h \in H. \tag{29}$$

Because $h \cdot \eta_M(m) = (\mathrm{Ad}_h \eta)_M(m) \in T_m(G_\mu \cdot m)$ and, by part (ii), W is H-invariant, formula (29) implies that $h \cdot w' = w'$ for all $h \in H$, that is, $w' \in W^H \subset T_m \boldsymbol{K}_L^{-1}(\lambda_\circ)$, and that $h \cdot \eta_M(m) = \eta_M(m)$ for all $h \in H$. This latter equality is equivalent to $\mathrm{Ad}_h \eta - \eta \in \mathfrak{h}$ for all $h \in H$. At this point we need the following statement.

Lemma 5.4. *Let K be a Lie group and H be a closed subgroup with corresponding Lie algebras $\mathfrak{k}$ and $\mathfrak{h}$. If $\xi \in \mathfrak{k}$ satisfies*

$$\mathrm{Ad}_h \xi - \xi \in \mathfrak{h}$$

for all $h \in H$, then ξ lies in the Lie algebra of the normalizer $N(H)$ of H.

Proof. Suppose that $\mathrm{Ad}_h \xi - \xi \in \mathfrak{h}$ for all $h \in H$. By taking the derivative relative to h at the identity, it follows that $[\eta, \xi] \in \mathfrak{h}$ for all $\eta \in \mathfrak{h}$. By the Baker–Campbell–Hausdorff formula (see, for example, [KMS93], p 40), $\exp(t \mathrm{Ad}_h \xi) \exp(-t\xi) = \exp(t(\mathrm{Ad}_h \xi - \xi) + \mathrm{O}(t^2))$, where $\mathrm{O}(t^2)$ is a convergent series each of whose terms is some iterated bracket of $\mathrm{Ad}_h \xi$ and ξ in some order, but always applied to $[\mathrm{Ad}_h \xi, \xi] = [\mathrm{Ad}_h \xi - \xi, \xi] \in \mathfrak{h}$ since $\mathrm{Ad}_h \xi - \xi \in \mathfrak{h}$. Thus each time one takes a bracket with ξ the result is in $\mathfrak{h}$ and each time one takes the bracket with $\mathrm{Ad}_h \xi$, one adds and subtracts a ξ to get two terms: the first, a bracket with ξ which lies in $\mathfrak{h}$, the second, a bracket with $\mathrm{Ad}_h \xi - \xi \in \mathfrak{h}$, which again lies in $\mathfrak{h}$, because both elements in the bracket are in $\mathfrak{h}$. The conclusion is that each term in this series lies in $\mathfrak{h}$ and hence $h \exp(t\xi) h^{-1} \exp(-t\xi) = \exp(t \mathrm{Ad}_h \xi) \exp(-t\xi) = \exp(t(\mathrm{Ad}_h \xi - \xi) + \mathrm{O}(t^2)) \in H$ for all $h \in H$ and all $t \in \mathbb{R}$. Therefore $\exp(t\xi) H \exp(-t\xi) \subseteq H$ for all $t \in \mathbb{R}$ which says that $\exp(t\xi) \in N(H)$ for all $t \in \mathbb{R}$, that is, ξ is in the Lie algebra of $N(H)$. This proves lemma 5.4. □

Returning to the main proof, by applying lemma 5.4 we conclude that η lies in the Lie algebra of $N(H)$. Therefore, $\eta_M(m) \in T_m(N_{G_\mu}(H) \cdot m) = T_m(L_{\lambda_\circ} \cdot m)$ so there exists some $\nu \in \mathfrak{l}_{\lambda_\circ}$ such that $\eta_M(m) = \nu_M(m)$. Consequently, by the definition of Δ, we have $\Delta(w') = w + T_m(L_{\lambda_\circ} \cdot m)$. Δ is hence surjective.

Finally, we prove that Δ is a symplectic linear map. Let $[v]_{\lambda_\circ} := v + T_m(L_{\lambda_\circ} \cdot m)$ and $[w]_{\lambda_\circ} := w + T_m(L_{\lambda_\circ} \cdot m)$ be arbitrary elements in $T_m \boldsymbol{K}_L^{-1}(\lambda_\circ)/T_m(L_{\lambda_\circ} \cdot m) \simeq T_{[m]_{\lambda_\circ}}(\boldsymbol{K}_L^{-1}(\lambda_\circ)/L_{\lambda_\circ})$. Recall from general reduction theory ([AM78], chapter 4), that via this isomorphism (which depends on the choice of the representative $m \in [m]_{\lambda_\circ}$) the quotient map $T_m \boldsymbol{K}_L^{-1}(\lambda_\circ) \to T_m \boldsymbol{K}_L^{-1}(\lambda_\circ)/T_m(L_{\lambda_\circ} \cdot m)$ is identified with $T_m \pi_{\lambda_0}$. The definition of Δ shows that it is the restriction of this quotient map to W^H. In particular, since Δ is surjective, v and w can always be chosen in W^H. The definition of the reduced symplectic form ω_{λ_0} implies then that

$$\omega_{\lambda_\circ}([m]_{\lambda_\circ})(\Delta(v), \Delta(w)) = \omega_{\lambda_\circ}(\pi_{\lambda_\circ}(m))(T_m \pi_{\lambda_\circ}(v), T_m \pi_{\lambda_\circ}(w)) = \omega(m)(v, w)$$

which shows that Δ is symplectic. □

Definition 5.5. *The subspace $W = \ker T_m \boldsymbol{J} \cap T_m(G_\mu \cdot m)^\perp$ is called the* orthosymplectic subspace *through $m \in M$ associated to the H-invariant metric $\langle \cdot, \cdot \rangle$.*

We now state the first main result of this section.

Theorem 5.6. *Assume the hypotheses and notations of theorem 4.3. Let W be the orthosymplectic subspace through $m \in M$, associated to some H-invariant metric $g = \langle \cdot, \cdot \rangle$.*

Then, the restriction $\boldsymbol{d}^2(h-\boldsymbol{J}^\xi)(m)|_{W\times W}$ of the Hessian of the augmented Hamiltonian $h-\boldsymbol{J}^\xi$ to W has the form

$$\boldsymbol{d}^2(h-\boldsymbol{J}^\xi)(m)|_{W\times W} = \begin{pmatrix} \boldsymbol{d}^2(h-\boldsymbol{J}^\xi)(m)|_{W^H\times W^H} & & 0 & \\ & A_1 & \cdots & 0 \\ 0 & \vdots & \ddots & \vdots \\ & 0 & \cdots & A_r \end{pmatrix}, \tag{30}$$

where $A_1, \ldots, A_r$ are the restrictions of $\boldsymbol{d}^2(h-\boldsymbol{J}^\xi)(m)$ to the isotypic components $W_1, \ldots, W_r$ of $(W^H)^{\omega(m)|_W}$.

Proof. We prove first that all the elements in the statement are well-defined. By proposition 5.3, the orthosymplectic space W is a Hamiltonian H-space and, by proposition 2.1, the set of H-fixed vectors W^H of W, is a symplectic subspace of W. Thus we can write the following symplectic direct sum

$$W = W^H \oplus (W^H)^{\omega(m)|_W}. \tag{31}$$

Both summands of (31) are H-invariant and W^H is the trivial isotypic component of W. Since $(W^H)^{\omega(m)|_W}$ is a H-invariant complementary subspace of W^H, it contains the remaining isotypic components of W. Therefore

$$W = W^H \oplus W_1 \oplus \cdots \oplus W_r \qquad \text{and} \qquad (W^H)^{\omega(m)|_W} = W_1 \oplus \cdots \oplus W_r, \tag{32}$$

are the isotypic decompositions of W and $(W^H)^{\omega(m)|_W}$, respectively.

We now prove the H-invariance of $\boldsymbol{d}^2(h-\boldsymbol{J}^\xi)(m)|_{W\times W}$. First, note that the G-invariance of h trivially implies its H-invariance. Secondly, recall that ξ is an orthogonal velocity of m relative to some Ad_H-invariant Riemannian metric. Lemma 4.4 guarantees then that $\mathrm{Ad}_h\xi = \xi$ for all $h \in H$. Therefore, if $z \in M$ and $h \in H$ are arbitrary, we have

$$\boldsymbol{J}^\xi(h\cdot z) = \langle \boldsymbol{J}(h\cdot z), \xi\rangle = \langle \mathrm{Ad}^*_{h^{-1}}\boldsymbol{J}(z), \xi\rangle = \langle \boldsymbol{J}(z), \mathrm{Ad}_{h^{-1}}\xi\rangle = \langle \boldsymbol{J}(z), \xi\rangle = \boldsymbol{J}^\xi(z).$$

Thirdly, Hessians commute with pull-backs. Thus, for $v, w \in W$ and $h \in H$, we have

$$\begin{aligned} \boldsymbol{d}^2(h-\boldsymbol{J}^\xi)(m)(h\cdot v, h\cdot w) &= (\phi_h^*(\boldsymbol{d}^2(h-\boldsymbol{J}^\xi)(m)))(v, w) \\ &= \boldsymbol{d}^2(\phi_h^* h - \phi_h^*\boldsymbol{J}^\xi)(h\cdot m)(v, w) \\ &= \boldsymbol{d}^2(h-\boldsymbol{J}^\xi)(m)(v, w), \end{aligned}$$

since $m \in M_H$; ϕ_h denotes both the H-action on M (in the second equality) and its linearization on W (in the first equality). Thus, $\boldsymbol{d}^2(h-\boldsymbol{J}^\xi)(m)|_{W\times W}$ is indeed H-invariant.

Corollary 5.2 implies the block diagonal form (30). □

The following theorem characterizes the role played by the Lyapunov stability of the reduced equilibrium $[m]_\mu^{(H)} \in M_\mu^{(H)}$ in the symplectic stratum, in the G_μ-stability analysis of the relative equilibrium $m \in M$.

Theorem 5.7. *In the hypotheses and notations of theorem 4.3, if $T_{[m]_\mu^{(H)}} M_\mu^{(H)}$ is identified with W^H via the natural symplectomorphism in proposition 5.3, then the* $(1, 1)$*-block in (30)* $\boldsymbol{d}^2(h-\boldsymbol{J}^\xi)(m)|_{W^H\times W^H} = \boldsymbol{d}^2 h_\mu^{(H)}([m]_\mu^{(H)})$*, where $\boldsymbol{d}^2 h_\mu^{(H)}([m]_\mu^{(H)})$ is the stability form associated to the equilibrium $[m]_\mu^{(H)} \in M_\mu^{(H)}$.*

Proof. As in proposition 5.3 (iii) we make the simplifying assumption that the normalizer $N(H)$ is compact. Recall that by theorem 2.6, the symplectic strata $(M_\mu^{(H)}, \omega_\mu^{(H)})$ and $(\boldsymbol{K}_L^{-1}(\lambda_\circ)/L_{\lambda_\circ}, \omega_{\lambda_\circ})$ are naturally symplectomorphic. We shall identify from now on these two symplectic manifolds. Thus, the proof of the Theorem is completed if we show that $\boldsymbol{d}^2(h - \boldsymbol{J}^\xi)(m)|_{W^H \times W^H} = \boldsymbol{d}^2 h_\mu^{(H)}([m]_\mu^{(H)})$, via the identification of

$$T_{[m]_\mu^{(H)}} M_\mu^{(H)} \simeq T_m \boldsymbol{K}_L^{-1}(\lambda_0)/T_m(L_{\lambda_0} \cdot m)$$

with W^H. Recall from proposition 5.3 (iii) that the quotient map

$$T_m \boldsymbol{K}_L^{-1}(\lambda_0) \to T_m \boldsymbol{K}_L^{-1}(\lambda_0)/T_m(L_{\lambda_0} \cdot m)$$

restricted to W^H is an isomorphism and hence we have

$$T_{[m]_\mu^{(H)}} M_\mu^{(H)} = \{T_m \pi_\mu^{(H)} \cdot v | v \in W^H\}.$$

We will now prove the following technical result.

Lemma 5.8. *Any $v \in W^H$ can be expressed as $v = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} F_t^v(m)$, with F_t^v the Hamiltonian flow of a $N(H)$-invariant function $g_v \in C^\infty(M)$.*

Proof. Recall from proposition 3.6 (MGS normal form for M_H) in section 3 that the symplectic manifold M_H is locally L-equivariantly symplectomorphic around the relative equilibrium $m \in M_H$ to $Y_H := L \times \mathfrak{l}_{\lambda_0}^* \times V_L$, where $V_L := \ker T_m \boldsymbol{K}_L / T_m(L_{\lambda_0} \cdot m)$ is the symplectic normal space at m to the orbit $L \cdot m$, $\mathfrak{l}_{\lambda_0}$ is the coadjoint isotropy of L at $\lambda_0 = \boldsymbol{K}_L(m) \in \mathfrak{l}^*$, and $\mathfrak{l} = \mathfrak{l}_{\lambda_0} \oplus \mathfrak{s}$ is an orthogonal decomposition relative to an $\mathrm{Ad}_{N(H)}$-invariant inner product on $\mathfrak{l}$. The point $m \in M_H$ corresponds under this symplectomorphism to $(e, 0, 0) \in Y_H$.

Let $v \in W^H = [\ker T_m \boldsymbol{J} \cap T_m(G_\mu \cdot m)^\perp]^H = \ker T_m \boldsymbol{J} \cap T_m M_H \cap [T_m(G_\mu \cdot m)]^\perp = \ker T_m \boldsymbol{K}_L \cap [T_m(G_\mu \cdot m)]^\perp$. We can replace a L-invariant neighbourhood of $m \in M$ with a symplectomorphic L-invariant neighbourhood of $(e, 0, 0) \in Y_H$. Under the derivative of this symplectomorphism, the tangent space $T_m M_H$ corresponds to $\mathfrak{l}_{\lambda_\circ} \times \mathfrak{s} \times \mathfrak{l}_{\lambda_\circ}^* \times V_L$, the momentum map $\boldsymbol{K}_L : M_H \to \mathfrak{l}^*$ to $\boldsymbol{K}_{L_{Y_H}} : (g, \eta, v) \in Y_H \mapsto g \cdot (\lambda_0 + \eta) \in \mathfrak{l}^*$, and the vector $v \in W^H$ to $(\rho, 0, 0, u)$ for some $\rho \in \mathfrak{l}_{\lambda_\circ}$ and $u \in V_L$. Equations (7) and (8) and the reconstruction method of the flow in section 5, guarantee that on taking a $N(H)$-invariant function $g_{V_L}^H$ on Y_H such that $D_{\mathfrak{l}_{\lambda_\circ}^*} g_{V_L}^H(e, 0, 0) = \rho$ and $D_{V_L} g_{V_L}^H(e, 0, 0) = \omega_{V_L}(u, \cdot)$, whose existence may readily be verified, the flow $F_t^{g_{V_L}^H}$ of the Hamiltonian vector field on Y_H defined by $g_{V_L}^H$, satisfies

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} F_t^{g_{V_L}^H}(e, 0, 0) = (\rho, 0, 0, u) \simeq v.$$

However, we need a flow satisfying this condition but whose Hamiltonian function is defined on the entire manifold M and is $N(H)$-invariant. First, we induce via the local symplectomorphism around m between M_H and Y_H a $N(H)$-invariant function on a $N(H)$-invariant neighbourhood of m in M_H and then use proposition 2.4 to extend this function to a $N(H)$-invariant function on M_H. Second, using again proposition 2.4, we extend this function to a $N(H)$-invariant function g_{V_L} on M. Let $g_{V_L}|_{M_H}$ denote the restriction of this function to M_H. By the previous relation, the flow of the Hamiltonian vector field of $g_{V_L}|_{M_H}$ on M_H satisfies

$$X_{g_{V_L}|_{M_H}}(m) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} F_t^{g_{V_L}|_{M_H}}(m) = v.$$

On the other hand, by $N(H)$-invariance of g_{V_L} on M, $F_t^{g_{V_L}}$ satisfies

$$F_t^{g_{V_L}} \circ \phi_n = \phi_n \circ F_t^{g_{V_L}} \qquad \text{for any} \quad n \in N(H),$$

where ϕ_n denotes the $N(H)$-action on M. This guarantees that $F_t^{g_{V_L}}(m) \in M^H$. Since M_H is an open subset of M^H and $m \in M_H$, by continuity, there is a $t_\circ > 0$ such that

$$F_t^{g_{V_L}}(m) \in M_H \qquad \text{whenever} \quad t < t_\circ.$$

This relation, the symplectic character of the natural inclusion $i : M_H \hookrightarrow M$, and $g_{V_L}|_{M_H} = g_{V_L} \circ i$, imply

$$T_m i \cdot X_{g_{V_L}|_{M_H}}(m) = X_{g_{V_L}}(i(m))$$

and hence

$$X_{g_{V_L}}(m) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} F_t^{g_{V_L}}(m) = T_m i \cdot X_{g_{V_L}|_{M_H}}(m) = v$$

as required. This proves lemma 5.8. □

The entries of the (1, 1)-block that we want to compute have the expressions

$$\boldsymbol{d}^2(h - \boldsymbol{J}^\xi)(m)(v, w), \qquad \text{for arbitrary} \qquad v, w \in W^H.$$

By lemma 5.8, there is a $N(H)$-invariant function g_v on M, whose Hamiltonian flow F_t^v satisfies $v = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} F_t^v(m)$. We extend w to a vector field $\mathcal{W}$ along $F_t^v(m)$ by setting

$$\mathcal{W}(F_t^v(m)) = T_m F_t^v \cdot w.$$

By the definition of the Hessian we get

$$\boldsymbol{d}^2(h - \boldsymbol{J}^\xi)(m)(v, w) = v[\mathcal{W}[h - \boldsymbol{J}^\xi]] = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathcal{W}[h - \boldsymbol{J}^\xi](F_t^v(m))$$
$$= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\boldsymbol{d}h(F_t^v(m)) \cdot T_m F_t^v \cdot w - \boldsymbol{dJ}^\xi(F_t^v(m)) \cdot T_m F_t^v \cdot w).$$

We compute the two summands of this expression separately. The second term equals:

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \boldsymbol{dJ}^\xi(F_t^v(m)) \cdot T_m F_t^v \cdot w = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \omega(F_t^v(m))(X_{\boldsymbol{J}^\xi}(F_t^v(m)), T_m F_t^v \cdot w)$$
$$= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \omega(F_t^v(m))(\xi_M(F_t^v(m)), T_m F_t^v \cdot w). \tag{33}$$

Since $\xi \in \mathrm{Lie}(N_{G_\mu}(H))$ and F_t^v is $N(H)$-equivariant,

$$\xi_M(F_t^v(m)) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \exp t\xi \cdot F_t^v(m) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} F_t^v(\exp t\xi \cdot m) = T_m F_t^v \cdot \xi_M(m)$$

hence the expression (33) equals

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \omega(F_t^v(m))(T_m F_t^v \cdot \xi_M(m), T_m F_t^v \cdot w) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} ((F_t^v)^*\omega)(m)(\xi_M(m), w)$$
$$= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \omega(m)(\xi_M(m), w) = 0$$

because F_t^v is a symplectomorphism. Thus the second term vanishes. Hence,

$$\boldsymbol{d}^2(h - \boldsymbol{J}^\xi)(m)(v, w) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \boldsymbol{d}h(F_t^v(m)) \cdot T_m F_t^v \cdot w. \tag{34}$$

We will use this identity to prove that

$$\boldsymbol{d}^2(h - \boldsymbol{J}^\xi)(m)(v, w) = \boldsymbol{d}^2 h_\mu^{(H)}([m]_\mu^{(H)})(T_m\pi_\mu^{(H)} \cdot v, T_m\pi_\mu^{(H)} \cdot w) \qquad \text{for all} \quad v, w \in W^H.$$

Recall that $h_\mu^{(H)}$ is defined by the relation $h_\mu^{(H)} \circ \pi_\mu^{(H)} = h \circ i_\mu^{(H)}$, where, if we identify $(M_\mu^{(H)}, \omega_\mu^{(H)})$ with $(\boldsymbol{K}_L^{-1}(\lambda_\circ)/L_{\lambda_\circ}, \omega_{\lambda_\circ})$, we can take for $\pi_\mu^{(H)}$ and $i_\mu^{(H)}$ (abusing the notation) the canonical projection $\pi_\mu^{(H)} : \boldsymbol{K}_L^{-1}(\lambda_\circ) \to \boldsymbol{K}_L^{-1}(\lambda_\circ)/L_{\lambda_\circ}$ and immersion $i_\mu^{(H)} : \boldsymbol{K}_L^{-1}(\lambda_\circ) \hookrightarrow M$.

The vector $T_m\pi_\mu^{(H)} \cdot w \in T_{[m]_\mu^{(H)}} M_\mu^{(H)}$ can be extended, using F_t^v, to a vector field $\mathcal{W}_\mu^{(H)}$ along $\pi_\mu^{(H)}(F_t^v(m))$ by

$$\mathcal{W}_\mu^{(H)}(\pi_\mu^{(H)}(F_t^v(m))) = T_m(\pi_\mu^{(H)} \circ F_t^v) \cdot w.$$

Then, since $\boldsymbol{d}h_\mu^{(H)}([m]_\mu^{(H)}) = 0$, we get

$$\begin{aligned} \boldsymbol{d}^2 h_\mu^{(H)}([m]_\mu^{(H)})(T_m\pi_\mu^{(H)} \cdot v, T_m\pi_\mu^{(H)} \cdot w) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \boldsymbol{d}h_\mu^{(H)}((\pi_\mu^{(H)} \circ F_t^v)(m))(T_m(\pi_\mu^{(H)} \circ F_t^v) \cdot w) \\ &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \boldsymbol{d}(h_\mu^{(H)} \circ \pi_\mu^{(H)})(F_t^v(m)) \cdot T_m F_t^v \cdot w \\ &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \boldsymbol{d}(h \circ i_\mu^{(H)})(F_t^v(m)) \cdot T_m F_t^v \cdot w. \end{aligned}$$

which coincides with expression (34), proving our claim. □

Summary of the method. We have shown that taking the orthosymplectic subspace

$$W = \ker T_m\boldsymbol{J} \cap T_m(G_\mu \cdot m)^\perp$$

constructed with a H-invariant metric, the relative equilibrium m is G_μ-stable if the symmetric matrix

$$\begin{pmatrix} \boldsymbol{d}^2(h - \boldsymbol{J}^\xi)(m) = |_{W^H \times W^H} & 0 \\ 0 & \begin{matrix} A_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & A_r \end{matrix} \end{pmatrix}, \tag{35}$$

is definite, where $A_1, \ldots, A_r$ are the restrictions of $\boldsymbol{d}^2(h - \boldsymbol{J}^\xi)(m)$ to the isotypic components $W_1, \ldots, W_r$ of the H-space $(W^H)^{\omega(m)|_W}$, with ξ an orthogonal velocity of m with respect to certain Ad_H-invariant splitting of $\mathrm{Lie}(N_{G_\mu}(H))$. In addition, relative to the natural symplectomorphism in proposition 5.3 (iii), the (1, 1)-block is given by

$$\boldsymbol{d}^2(h - \boldsymbol{J}^\xi)(m)|_{W^H \times W^H} = \boldsymbol{d}^2 h_\mu^{(H)}([m]_\mu^{(H)}), \tag{36}$$

where $\boldsymbol{d}^2 h_\mu^{(H)}([m]_\mu^{(H)})$ is the stability form associated to the reduced equilibrium $[m]_\mu^{(H)}$ on the stratum $M_\mu^{(H)}$.

To sum up, given a relative equilibrium $m \in M$, theorems 4.3, 5.6, and 5.7 guarantee that m is stable modulo G_μ if the following three conditions are satisfied:

(i) The bilinear form $\boldsymbol{d}^2 h_\mu^{(H)}([m]_\mu^{(H)})$ is definite and, therefore, the associated singular reduced equilibrium is Lyapunov stable in its stratum;

(ii) The bilinear forms $\boldsymbol{d}^2(h - \boldsymbol{J}^\xi)(m)|_{W_i \times W_i}$ are definite, for any $i \in \{1, \ldots, r\}$, with ξ an orthogonal velocity of m;

(iii) All the definite bilinear forms in (i) and (ii) have the same sign.

Finally, as we have already remarked, the orthogonal velocity of a relative equilibrium is only uniquely determined if the Ad_H-invariant inner product on $\mathrm{Lie}(N_{G_\mu}(H))$ is specified. Since there are many such inner products, in analysing the G_μ-stability of the relative equilibrium m one has to optimize over all such choices. This needs to be done on a case by case basis. We will carry this out explicitly in the example below.

6. An example: the stability of the sleeping Lagrange top

One of the simplest systems that exhibits singular relative equilibria is the Lagrange top in the upright position, that is, an axisymmetric rigid body with a fixed point moving steadily in a gravitational field in such a fashion that the axis of symmetry, the centre-of-mass vector, and the axis of gravity are all parallel.

The stability of this relative equilibrium is a classical result in elementary mechanics. See [Hal85] for its derivation using the energy–Casimir method and [Lal92] for its derivation using the reduced energy–momentum method and its spectral stability analysis. We will show below that using the results presented in the previous sections, the same classical optimal stability result can be obtained. This example is intended only as an illustration of how the techniques in this paper can be applied to singular relative equilibria.

The notation used here is identical to the one in [Lal92]. In particular, the elements of $TSO(3)$ in spatial representation (that is, right trivialization) will be expressed as $(\Lambda, \widehat{\boldsymbol{\delta\theta}}\Lambda)$ where $\Lambda \in SO(3)$ and $\widehat{\boldsymbol{\delta\theta}}$ is the skew-symmetric matrix associated to $\boldsymbol{\delta\theta} \in \mathbb{R}^3$ via the relation $\widehat{\boldsymbol{\delta\theta}}\boldsymbol{x} = \boldsymbol{\delta\theta} \times \boldsymbol{x}$. Analogously, the elements of $T^*SO(3)$ have the form $(\Lambda, \widehat{\boldsymbol{\pi}}\Lambda)$ with $\boldsymbol{\pi} \in \mathbb{R}^3$. The symbol $\boldsymbol{g} = g\boldsymbol{e_3}$ denotes the gravity vector, where $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ is a spatial orthonormal basis of $\mathbb{R}^3$. We define the mass vector by $\boldsymbol{M} := \int_{\mathcal{B}} \rho_{ref}(X) X \mathrm{d}^3 X$, where $\mathcal{B}$ is a reference configuration. If m is the total mass of the body and l is the distance from the fixed point to the centre-of-mass, then $|\boldsymbol{M}| = ml$. The *reference inertia tensor* $\mathbb{I}_{ref}$ is defined as

$$\mathbb{I}_{ref} := \int_{\mathcal{B}} \rho_{ref}(X)(|X|^2 \mathbb{I}_3 - X \otimes X)\, \mathrm{d}^3 X$$

and the *current spatial inertia tensor* is given by

$$\mathbb{I}_\Lambda := \Lambda \mathbb{I}_{ref} \Lambda^T.$$

If $\boldsymbol{m} := \Lambda \boldsymbol{M}$ is the spatial representation of the mass vector, in these variables, the Hamiltonian of the heavy top is given by

$$h(\Lambda, \boldsymbol{\pi}) := \boldsymbol{m} \cdot \boldsymbol{g} + \tfrac{1}{2} \boldsymbol{\pi} \cdot \mathbb{I}_\Lambda^{-1} \boldsymbol{\pi}$$

and its Lagrangian is

$$L(\Lambda, \boldsymbol{\omega}) := \tfrac{1}{2} \boldsymbol{\omega} \cdot \mathbb{I}_\Lambda \boldsymbol{\omega} - \boldsymbol{g} \cdot \Lambda \boldsymbol{M}.$$

The symmetry properties of the Lagrange top allow us to choose, without loss of generality, $\mathbb{I}_{ref} = \mathrm{diag}[I_1, I_1, I_3]$ for some constants I_1 and I_3. The reference configuration will be chosen in such a way that the axis of symmetry of the top will be parallel to the gravity vector, that is, $\boldsymbol{M} = ml\boldsymbol{e_3}$.

The symmetries of this system are given by the Hamiltonian action of the Abelian Lie group $G = S^1 \times S^1$ on the phase space of the system, that is, $T^*SO(3)$. Using spatial variables, the G-action on the space of velocities has the form

$$G \times TSO(3) \longrightarrow TSO(3)$$
$$((\theta_1, \theta_2), (\Lambda, \boldsymbol{\delta\theta})) \longmapsto (\exp(\theta_1 \widehat{\boldsymbol{e_3}}) \Lambda \exp(-\theta_2 \widehat{\boldsymbol{e_3}}), \exp(\theta_1 \widehat{\boldsymbol{e_3}}) \boldsymbol{\delta\theta})$$

and on the phase space

$$G \times T^*SO(3) \longrightarrow T^*SO(3)$$
$$((\theta_1, \theta_2), (\Lambda, \boldsymbol{\pi})) \longmapsto (\exp(\theta_1 \widehat{\boldsymbol{e_3}}) \Lambda \exp(-\theta_2 \widehat{\boldsymbol{e_3}}), \exp(-\theta_1 \widehat{\boldsymbol{e_3}}) \boldsymbol{\pi}).$$

The infinitesimal generators associated to the G-action on the configuration space $Q = SO(3)$ are given by

$$(\xi, \omega)_Q(\Lambda) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \exp(t\xi \widehat{\boldsymbol{e_3}}) \Lambda \exp(-t\omega \widehat{\boldsymbol{e_3}}) = \xi \widehat{\boldsymbol{e_3}} \Lambda - \omega \Lambda \widehat{\boldsymbol{e_3}}$$
$$= \xi \widehat{\boldsymbol{e_3}} \Lambda - \omega \Lambda \widehat{\boldsymbol{e_3}} \Lambda^T \Lambda = (\xi \widehat{\boldsymbol{e_3}} - \omega \widehat{\Lambda \boldsymbol{e_3}}) \Lambda,$$

for some $\xi, \omega \in \mathrm{Lie}(\mathrm{S}^1) = \mathbb{R}$. We hence have that, in spatial coordinates

$$(\xi, \omega)_Q(\Lambda) = (\Lambda, \xi \boldsymbol{e_3} - \omega \Lambda \boldsymbol{e_3}).$$

In the same spatial coordinates, the momentum map $\boldsymbol{J} : T^*SO(3) \to \mathfrak{g}^* \simeq \mathbb{R}^2$, corresponding to the lifted action is given by

$$\langle \boldsymbol{J}(\Lambda, \boldsymbol{\pi}), (\xi, \omega)\rangle = \langle (\Lambda, \boldsymbol{\pi}), (\Lambda, \xi \boldsymbol{e_3} - \omega \Lambda \boldsymbol{e_3})\rangle = \xi \boldsymbol{\pi} \cdot \boldsymbol{e_3} - \omega \boldsymbol{\pi} \cdot \Lambda \boldsymbol{e_3},$$

hence

$$\boldsymbol{J}(\Lambda, \boldsymbol{\pi}) = (\boldsymbol{\pi} \cdot \boldsymbol{e_3}, -\boldsymbol{\pi} \cdot \Lambda \boldsymbol{e_3}).$$

We now show how any sleeping top is a relative equilibrium, in other words, every point in $T^*SO(3)$ of the form $\mathbb{F}L((\xi, \omega)_Q(I))$, for certain $(\xi, \omega) \in \mathfrak{g}$ and I the identity element of $SO(3)$, is a relative equilibrium. The symbol $\mathbb{F}L$ denotes the fibre derivative of the function L which is a vector bundle map $TSO(3) \to T^*SO(3)$. If we take $z = \mathbb{F}L((\xi, \omega)_Q(I)) = (I, (\xi - \omega) I_3 \boldsymbol{e_3}) := (I, \boldsymbol{\pi_e})$, by theorem 2.8, z is a relative equilibrium iff there is an element $(\alpha_1, \alpha_2) \in \mathfrak{g} = \mathbb{R}^2$ for which

$$\boldsymbol{d}(h - \boldsymbol{J}^{(\alpha_1, \alpha_2)})(z) = 0.$$

Since $\boldsymbol{J}^{(\alpha_1, \alpha_2)}(\Lambda, \boldsymbol{\pi}) = \alpha_1 \boldsymbol{\pi} \cdot \boldsymbol{e_3} - \alpha_2 \boldsymbol{\pi} \cdot \Lambda \boldsymbol{e_3}$, we get

$$\boldsymbol{dJ}^{(\alpha_1, \alpha_2)}(\Lambda, \boldsymbol{\pi})(\delta\Lambda, \boldsymbol{\delta\pi}) = \alpha_1 \boldsymbol{\delta\pi} \cdot \boldsymbol{e_3} - \alpha_2 \boldsymbol{\delta\pi} \cdot \Lambda \boldsymbol{e_3} - \alpha_2 \boldsymbol{\pi} \cdot \widehat{\boldsymbol{\delta\theta}} \Lambda \boldsymbol{e_3},$$

where $\delta\Lambda := \widehat{\boldsymbol{\delta\theta}}\Lambda$. Hence,

$$\boldsymbol{dJ}^{(\alpha_1, \alpha_2)}(z)(\delta\Lambda, \boldsymbol{\delta\pi}) = (\alpha_1 - \alpha_2)\boldsymbol{\delta\pi} \cdot \boldsymbol{e_3} - \alpha_2(\xi - \omega) I_3 \boldsymbol{e_3} \cdot \widehat{\boldsymbol{\delta\theta}} \boldsymbol{e_3} = (\alpha_1 - \alpha_2)\boldsymbol{\delta\pi} \cdot \boldsymbol{e_3},$$

where we have used the relation $\boldsymbol{e_3} \cdot \widehat{\boldsymbol{\delta\theta}} \boldsymbol{e_3} = \boldsymbol{e_3} \cdot (\boldsymbol{\delta\theta} \times \boldsymbol{e_3}) = 0$. On the other hand, since

$$h(\Lambda, \boldsymbol{\pi}) = mgl\boldsymbol{e_3} \cdot \Lambda \boldsymbol{e_3} + \tfrac{1}{2} \boldsymbol{\pi} \cdot \Lambda \mathbb{I}_{ref}^{-1} \Lambda^T \boldsymbol{\pi},$$

the first derivative is

$$\boldsymbol{d}h(\Lambda, \boldsymbol{\pi})(\delta\Lambda, \boldsymbol{\delta\pi}) = mgl\boldsymbol{e_3} \cdot \widehat{\boldsymbol{\delta\theta}} \Lambda \boldsymbol{e_3} + \boldsymbol{\delta\pi} \cdot \Lambda \mathbb{I}_{ref}^{-1} \Lambda^T \boldsymbol{\pi} + \boldsymbol{\pi} \cdot \widehat{\boldsymbol{\delta\theta}} \Lambda \mathbb{I}_{ref}^{-1} \Lambda^T \boldsymbol{\pi},$$

which evaluated at z gives

$$\boldsymbol{d}h(m)(\delta\Lambda, \boldsymbol{\delta\pi}) = (\xi - \omega)\boldsymbol{\delta\pi} \cdot \boldsymbol{e_3} + (\xi - \omega)^2 I_3 \boldsymbol{e_3} \cdot \widehat{\boldsymbol{\delta\theta}} \boldsymbol{e_3} = (\xi - \omega)\boldsymbol{\delta\pi} \cdot \boldsymbol{e_3}.$$

Hence the derivative of the augmented Hamiltonian equates to

$$\boldsymbol{d}(h - \boldsymbol{J}^{(\alpha_1, \alpha_2)})(z)(\delta\Lambda, \boldsymbol{\delta\pi}) = ((\xi - \omega) - (\alpha_1 - \alpha_2))\boldsymbol{\delta\pi} \cdot \boldsymbol{e_3}.$$

Therefore, in order to have $\boldsymbol{d}(h - \boldsymbol{J}^{(\alpha_1, \alpha_2)})(z) = 0$ and consequently to prove that z is a relative equilibrium we just need to take $(\alpha_1 - \alpha_2)$ such that $\lambda := (\xi - \omega) = (\alpha_1 - \alpha_2)$. Moreover, according to our general definition, (ξ, ω) is a velocity for the relative equilibrium z. This relative equilibrium will be called *upright sleeping top*. We will see that its stability depends on the parameter λ. A top whose tip points in the direction of the gravity vector is called a *hanging top* and the study of its stability is handled analogously to the upright sleeping case.

It should be immediately noticed that z has non-trivial symmetry. Indeed if $(\theta_1, \theta_2) \in G$ is such that $(\theta_1, \theta_2) \cdot z = z$, that is, $(\exp((\theta_1, \theta_2)\widehat{\boldsymbol{e_3}}), (\xi - \omega) I_3 \boldsymbol{e_3}) = (I, (\xi - \omega) I_3 \boldsymbol{e_3})$, then $\theta_1 = \theta_2$ and thus

$$H = \{(\theta_1, \theta_2) \in G | \theta_1 = \theta_2\}.$$

It is also easy to check that

$$(T^*SO(3))_H = \{(\exp(\psi \widehat{\boldsymbol{e_3}}), \pi \boldsymbol{e_3}) | \psi \in \mathrm{Lie}(S^1) = \mathbb{R}, \pi \in \mathbb{R}\}.$$

Let $\mu = \boldsymbol{J}(z) = ((\xi - \omega) I_3, -(\xi - \omega) I_3)$ be the momentum value of z. Since G is Abelian, $G_\mu = G$ and $N_{G_\mu}(H) = G$ hence, in this particular case, $L := N(H)/H = G/H$.

Now, using the notation introduced in section 2, it is easy to see by dimension count that the symplectic stratum

$$T^*SO(3)_\mu^{(H)} \simeq \boldsymbol{K}_L^{-1}(\lambda_\circ)/L_{\lambda_\circ} = \boldsymbol{K}_L^{-1}(\lambda_\circ)/(G/H)$$

is trivial. Therefore the (1, 1) block of the stability form vanishes.

We now determine $\ker T_z\boldsymbol{J}$. Let $(\delta\Lambda, \boldsymbol{\delta\pi}) \in T_z(T^*SO(3))$ be such that $\widehat{\delta\Lambda} = \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} \Lambda(t)$ and $\boldsymbol{\delta\pi} = \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} \boldsymbol{\pi}(t)$ with $\Lambda(0) = I$ and $\boldsymbol{\pi}(0) = (\xi - \omega)I_3\boldsymbol{e_3} = \boldsymbol{\pi}_e$. Then

$$\begin{aligned}(0,0) = T_z\boldsymbol{J}(\delta\Lambda, \boldsymbol{\delta\pi}) &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} (\boldsymbol{\pi}(t)\cdot\boldsymbol{e_3}, -\boldsymbol{\pi}(t)\cdot\Lambda(t)\boldsymbol{e_3}) \\ &= (\boldsymbol{\delta\pi}\cdot\boldsymbol{e_3}, -\boldsymbol{\delta\pi}\cdot\boldsymbol{e_3} - \boldsymbol{\pi}_e\cdot\widehat{\delta\Lambda}\boldsymbol{e_3}),\end{aligned}$$

which implies that

$$\boldsymbol{\delta\pi}\cdot\boldsymbol{e_3} = 0 \qquad \text{and} \qquad \boldsymbol{\pi}_e\cdot\widehat{\delta\Lambda}\boldsymbol{e_3} = 0. \tag{37}$$

However $\boldsymbol{\pi}_e\cdot\widehat{\delta\Lambda}\boldsymbol{e_3} = \boldsymbol{\pi}_e\cdot(\delta\Lambda\times\boldsymbol{e_3}) = 0$, hence the second expression in (37) does not impose any restriction on $\delta\Lambda$ and consequently

$$\ker T_z\boldsymbol{J} = \{(\delta\Lambda, \boldsymbol{\delta\pi}) \in T_z(T^*SO(3)) \mid \boldsymbol{\delta\pi}\cdot\boldsymbol{e_3} = 0\}.$$

One computes similarly

$$T_z(G_\mu\cdot z) = T_z(G\cdot z) = \mathrm{span}\{(\widehat{\boldsymbol{e_3}}, 0)\}.$$

Looking at the spaces that we have already computed, it seems that the most convenient metric that one can take to construct the orthosymplectic space is the one that gives us the natural Euclidean inner product in $T_z(T^*SO(3)) \simeq \mathbb{R}^3\times\mathbb{R}^3$, for which

$$W = \ker T_z\boldsymbol{J} \cap T_z(G\cdot z)^\perp = \{(\boldsymbol{\delta\Lambda}, \boldsymbol{\delta\pi}) \in T_z(T^*SO(3)) \mid \boldsymbol{\delta\Lambda}\cdot\boldsymbol{e_3} = \boldsymbol{\delta\pi}\cdot\boldsymbol{e_3} = 0\}.$$

It is easy to see that W is the direct sum of two bidimensional H-irreducible spaces. These two spaces are trivially H-isomorphic, therefore W has a trivial isotypic decomposition and, in principle there is no blocking of the kind predicted by theorem 5.1.

With all these ingredients, we compute the Hessian of the Lyapunov function at z, restricted to W. If $\delta\Lambda_1 = \widehat{\boldsymbol{\delta\theta}}_1\Lambda$ and $\delta\Lambda_2 = \widehat{\boldsymbol{\delta\theta}}_2\Lambda$ are elements in $T_\Lambda SO(3)$ we have

$$\begin{aligned}\boldsymbol{d}^2h(\Lambda,\boldsymbol{\pi})\cdot((\delta\Lambda_1,\boldsymbol{\delta\pi}_1),(\delta\Lambda_2,\boldsymbol{\delta\pi}_2)) &= mgl\boldsymbol{e_3}\cdot\widehat{\boldsymbol{\delta\theta}}_1\widehat{\boldsymbol{\delta\theta}}_2\Lambda\boldsymbol{e_3} + \boldsymbol{\delta\pi}_1\cdot\Lambda\mathbb{I}_{ref}^{-1}\Lambda^T\boldsymbol{\delta\pi}_2 \\ &+\boldsymbol{\delta\pi}_2\cdot\widehat{\boldsymbol{\delta\theta}}_1\Lambda\mathbb{I}_{ref}^{-1}\Lambda^T\boldsymbol{\pi} + \boldsymbol{\pi}\cdot\widehat{\boldsymbol{\delta\theta}}_1\Lambda\mathbb{I}_{ref}^{-1}\Lambda^T\boldsymbol{\delta\pi}_2 + \boldsymbol{\delta\pi}_1\cdot\widehat{\boldsymbol{\delta\theta}}_2\Lambda\mathbb{I}_{ref}^{-1}\Lambda^T\boldsymbol{\pi} \\ &-\boldsymbol{\delta\pi}_1\cdot\Lambda\mathbb{I}_{ref}^{-1}\Lambda^T\widehat{\boldsymbol{\delta\theta}}_2\boldsymbol{\pi} + \boldsymbol{\pi}\cdot\widehat{\boldsymbol{\delta\theta}}_1\widehat{\boldsymbol{\delta\theta}}_2\Lambda\mathbb{I}_{ref}^{-1}\Lambda^T\boldsymbol{\pi} - \boldsymbol{\pi}\cdot\widehat{\boldsymbol{\delta\theta}}_1\Lambda\mathbb{I}_{ref}^{-1}\Lambda^T\widehat{\boldsymbol{\delta\theta}}_2\boldsymbol{\pi}.\end{aligned}$$

We evaluate this expression at z, using the parameter $\lambda = \xi - \omega$. Since $\boldsymbol{\delta\theta}_i\cdot\boldsymbol{e_3} = \boldsymbol{\delta\pi}_i\cdot\boldsymbol{e_3} = 0$ we get that $\mathbb{I}_{ref}^{-1}\boldsymbol{\delta\pi}_i = \frac{1}{I_1}\boldsymbol{\delta\pi}_i$ and $\mathbb{I}_{ref}^{-1}\boldsymbol{\delta\theta}_i = \frac{1}{I_1}\boldsymbol{\delta\theta}_i$ for $i \in \{1, 2\}$. Therefore,

$$\begin{aligned}\boldsymbol{d}^2h(z)\cdot((\delta\Lambda_1,\boldsymbol{\delta\pi}_1),(\delta\Lambda_2,\boldsymbol{\delta\pi}_2)) &= mgl\boldsymbol{e_3}\cdot\widehat{\boldsymbol{\delta\theta}}_1\widehat{\boldsymbol{\delta\theta}}_2\boldsymbol{e_3} + \frac{1}{I_1}\boldsymbol{\delta\pi}_1\cdot\boldsymbol{\delta\pi}_2 + \lambda\boldsymbol{\delta\pi}_2\cdot\widehat{\boldsymbol{\delta\theta}}_1\boldsymbol{e_3} \\ &+\frac{\lambda I_3}{I_1}\boldsymbol{e_3}\cdot\widehat{\boldsymbol{\delta\theta}}_1\boldsymbol{\delta\pi}_2 + \lambda\boldsymbol{\delta\pi}_1\cdot\widehat{\boldsymbol{\delta\theta}}_2\boldsymbol{e_3} - \frac{\lambda I_3}{I_1}\boldsymbol{\delta\pi}_1\cdot\widehat{\boldsymbol{\delta\theta}}_2\boldsymbol{e_3} + \lambda^2I_3\boldsymbol{e_3}\cdot\widehat{\boldsymbol{\delta\theta}}_1\widehat{\boldsymbol{\delta\theta}}_2\boldsymbol{e_3} \\ &-\frac{\lambda^2I_3^2}{I_1}\boldsymbol{e_3}\cdot\widehat{\boldsymbol{\delta\theta}}_1\widehat{\boldsymbol{\delta\theta}}_2\boldsymbol{e_3}.\end{aligned} \tag{38}$$

In order to use theorem 4.3, we need to write the second summand of the augmented Hamiltonian using an orthogonal velocity, that is, the projection of (ξ, ω) on the orthogonal complement of $\mathfrak{h} = \mathrm{Lie}(H) = \mathrm{span}\{(1, 1)\}$ with respect to an Ad_G-invariant metric on $\mathfrak{g}$.

Since G is Abelian, any metric is Ad_G-invariant, hence the most general situation consists of taking the inner product in $\mathfrak{g}$ given by the quadratic form

$$g = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

subject to the condition $\det g = ac - b^2 > 0$, which ensures the positive definiteness of g. Using the notation introduced in section 4, the orthogonal complement $\mathfrak{p}_\mu$ of $\mathfrak{h}$ with respect to g is

$$\mathfrak{p}_\mu = \mathrm{span}\,\{(1, -k)\}$$

where $k = (a+b)/(b+c)$. This implies that the orthogonal velocity v_c of z with respect to the splitting determined by g is

$$v_c(k) = \lambda\left(\frac{1}{1+k}, \frac{-k}{1+k}\right).$$

At the same time it is easy to see that for $(\alpha_1, \alpha_2) \in \mathbb{R}^2$ arbitrary

$$\begin{aligned} &\boldsymbol{d}^2\boldsymbol{J}^{(\alpha_1,\alpha_2)}(\Lambda, \boldsymbol{\pi}) \cdot ((\delta\Lambda_1, \boldsymbol{\delta\pi}_1), (\delta\Lambda_2, \boldsymbol{\delta\pi}_2)) \\ &\qquad = -\alpha_2(\boldsymbol{\delta\pi}_1 \cdot \widehat{\boldsymbol{\delta\theta}}_2\Lambda\boldsymbol{e_3} - \boldsymbol{\delta\pi}_2 \cdot \widehat{\boldsymbol{\delta\theta}}_1\Lambda\boldsymbol{e_3} - \boldsymbol{\pi} \cdot \widehat{\boldsymbol{\delta\theta}}_1\widehat{\boldsymbol{\delta\theta}}_2\Lambda\boldsymbol{e_3}), \end{aligned}$$

which evaluated at z takes the form

$$\begin{aligned} &\boldsymbol{d}^2\boldsymbol{J}^{(\alpha_1,\alpha_2)}(z) \cdot ((\delta\Lambda_1, \boldsymbol{\delta\pi}_1), (\delta\Lambda_2, \boldsymbol{\delta\pi}_2)) \\ &\qquad = -\alpha_2(\boldsymbol{\delta\pi}_1 \cdot \widehat{\boldsymbol{\delta\theta}}_2\boldsymbol{e_3} - \boldsymbol{\delta\pi}_2 \cdot \widehat{\boldsymbol{\delta\theta}}_1\boldsymbol{e_3} - \lambda I_3\boldsymbol{e_3} \cdot \widehat{\boldsymbol{\delta\theta}}_1\widehat{\boldsymbol{\delta\theta}}_2\boldsymbol{e_3}). \end{aligned} \tag{39}$$

Now set $(\alpha_1, \alpha_2) = v_c(k)$ and combine the expressions (38) and (39) to obtain the following matrix of the Hessian $\boldsymbol{d}^2(h - \boldsymbol{J}^{v_c(k)})(z)$ restricted to W

$$\begin{pmatrix} -mgl - \lambda^2 I_3(\frac{1}{1+k} - \frac{I_3}{I_1}) & 0 & 0 & \lambda(\frac{I_3-I_1}{I_1} + \frac{k}{1+k}) \\ 0 & -mgl - \lambda^2 I_3(\frac{1}{1+k} - \frac{I_3}{I_1}) & -\lambda(\frac{I_3-I_1}{I_1} + \frac{k}{1+k}) & 0 \\ 0 & -\lambda(\frac{I_3-I_1}{I_1} + \frac{k}{1+k}) & \frac{1}{I_1} & 0 \\ \lambda(\frac{I_3-I_1}{I_1} + \frac{k}{1+k}) & 0 & 0 & \frac{1}{I_1} \end{pmatrix},$$

whose eigenvalues are

$$\sigma_\pm = A \pm \sqrt{-4I_1(1+k)^2B + A^2},$$

with

$$\begin{aligned} A &= (1+k)^2 - mglI_1(1+k)^2 + I_3\lambda^2(I_3(1+2k) - I_1(1+k)) \\ B &= \lambda^2(I_3k + I_3 - I_1) - mgl(1+k)^2. \end{aligned}$$

It is clear that $\boldsymbol{d}^2(h - \boldsymbol{J}^{v_c(k)})(z)$ is positive definite iff $B > 0$, that is

$$\lambda^2 > mgl\frac{(1+k)^2}{I_3k + I_3 - I_1},$$

hence for each k (for each orthogonal velocity) we have a lower bound for the values of λ for which the sleeping top is stable. The optimal stability condition will be achieved when

$$\frac{(1+k)^2}{I_3k + I_3 - I_1}$$

reaches a minimum. Taking the first and second derivatives of this function, one checks that this happens when

$$k = \frac{2I_1 - I_3}{I_3}$$

and therefore, the optimal stability condition is

$$\lambda^2 > \frac{4mglI_1^2}{I_3}, \tag{40}$$

which coincides with the classical one.

Note that

$$g = \begin{pmatrix} \frac{2I_1 - I_3}{I_3} & 0 \\ 0 & 1 \end{pmatrix}$$

is an inner product whose k is $\frac{2I_1 - I_3}{I_3}$ and it is positive definite because for any diagonal inertia tensor $\mathbb{I}_{ref} = \mathrm{diag}[\mathrm{I}_1, \mathrm{I}_1, \mathrm{I}_3]$, the inequalities

$$I_j + I_k > I_i \qquad \text{for all} \quad i, j, k \in \{1, 2, 3\}, \quad j \neq k$$

always hold. This guarantees the consistency of the optimal stability condition (40).

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