Feedback-driven recurrent quantum neural network universality

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Motivation

Quantum machine learning aims to exploit the potential significant computational speed-up and reduced complexity offered by quantum computing.

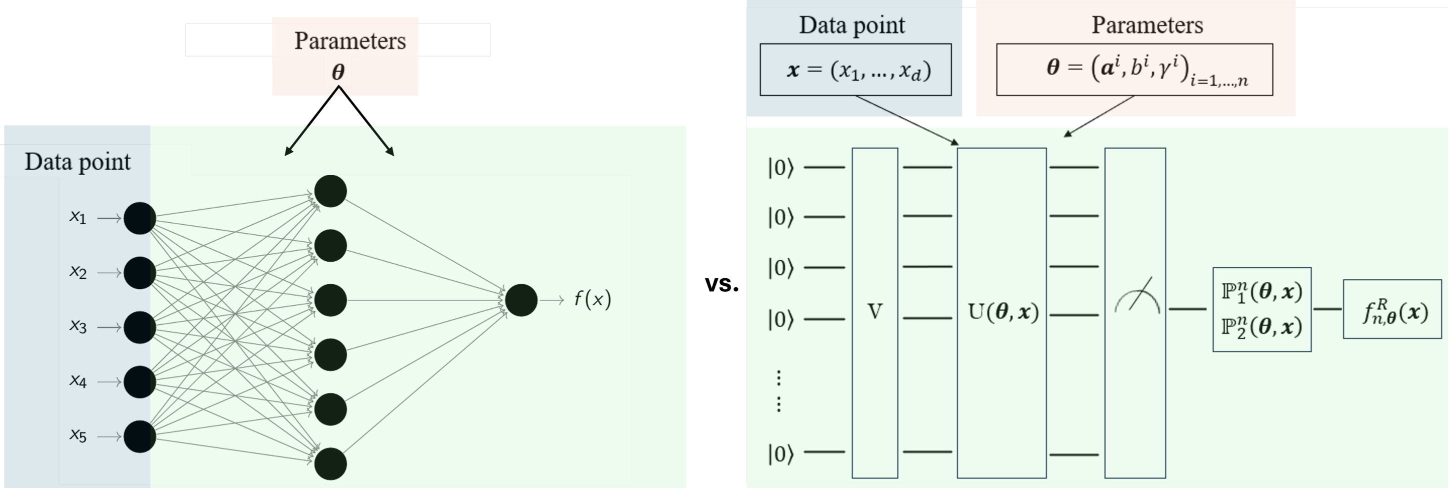

Figure 1. Classical vs. quantum neural networks

We study recurrent quantum neural networks (RQNNs):

- Quantum analogue of classical RNNs.
- Introduced for learning tasks involving time series data and temporal dynamics.
- RQNNs and related quantum reservoir computing methods show promising empirical performance , see, e.g., [1, 2].

Key questions:

1. Are recurrent quantum neural networks universal approximators for dynamical systems?
2. How many qubits are needed to achieve a given accuracy?

Recurrent Quantum Neural Network Architecture

RQNNs are realized via parametric quantum circuits with a feedback loop, visualized as follows:

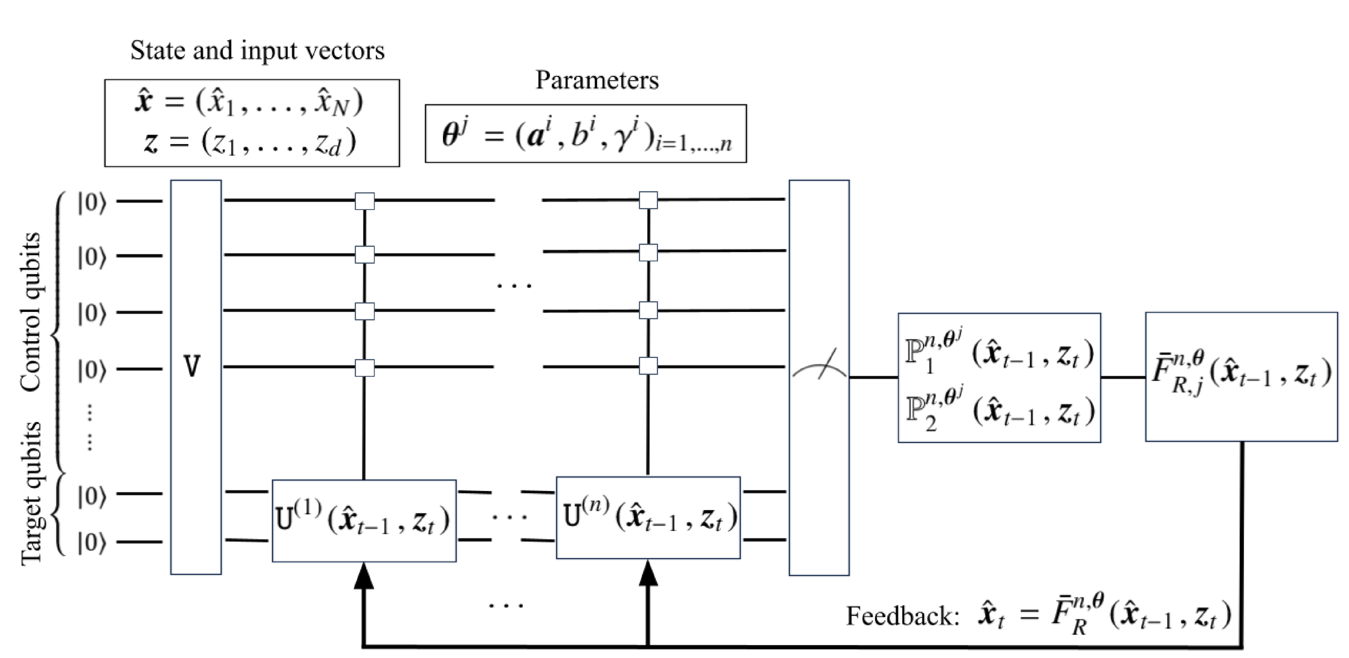

Figure 2. Schematic representation of the j-th circuit given by parameters $\theta^j \in, j \in \{1, \dots, N\}$.

All qubits are initialized from state $|0\rangle$ and transformed according to parametric quantum gates $\mathtt{V}$ and $\mathtt{U}_\theta(x, z)$.

At the end, we measure the state of each qubit. Running the circuit repeatedly, we infer output probabilities serving as hidden state feedback.

Mathematical QRNN setup

State space formulation: a QRNN transforms inputs according to

$$\hat{x}_t = \overline{F}_\theta(\hat{x}_{t-1}, z_t) \quad (1)$$
$$\mathbf{y}_t = W x_t \quad (2)$$

for $t \in \mathbb{Z}$ and with

$\hat{x}_t$	hidden state vector at time t
$\overline{F}_\theta$	RQNN circuit state update map
z_t	data at time t
$\mathbf{y}_t$	RQNN output vector at time t
W	readout matrix

The state map $\overline{F}_\theta$ consists of N parallel quantum circuits, whose measurement probabilities define the state update:

$$\overline{F}_{\theta,j}(x, z) := R - 2R[\mathbb{P}_{1,\theta^j}(x, z) + \mathbb{P}_{2,\theta^j}(x, z)].$$

$\mathbb{P}_{m,\theta^j}$ is the probability that the output state of the quantum circuit with parameters θ^j is in $\{m, 4+m, \dots, 4(n-1)+m\}$.

Feedforward QNN Approximation Rates

Gonon & Jacquier [3] established feedforward QNN approximation rates:

Approximation error: $\|f - f_n\| \le \frac{C}{\sqrt{n}}$ for all n

- for functions $f \in L^1(\mathbb{R}^d)$ with Fourier transform $\hat{f}$ and $C := \int |\hat{f}(\xi)| < \infty$,
- f_n a QNN with $\mathcal{O}(\log(n))$ qubits,
- $\|\cdot\|$ is the $L^2(\mu)$-norm or the L^∞-norm (+ further integrability).

Comparison to classical neural networks: same dimension-free approximation rate as in [4], but QNN bound holds for larger class of functions.

Building on these techniques, we show: RQNNs can approximate state-space maps and their derivatives.

If the state map F is C^1 with $B_j = \int_{\mathbb{R}^N \times \mathbb{R}^d} \max(1, \|\xi\|^2)|\widehat{F}_j(\xi)| d\xi < \infty$, then

$$\left\|\overline{F}_{\theta,j} - F_j\right\|^2 + \sum_{i=1}^{N+d} \left\|\partial_i \overline{F}_{\theta,j} - \partial_i F_j\right\|^2 \le \frac{(1+4\pi^2)B_j^2}{n} \quad \text{for all } n.$$

Main Result I: Dynamical System Approximation

Goal: approximate dynamical system (or state-space system)

$$x_t = F(x_{t-1}, z_t) \quad (3)$$

using a RQNN. Write $U(z)$ for associated filter: sequence-to-sequence map $U(z) = (\dots, x_{-1}(z), x_0(z), x_1(z), \dots)$ for $z = (\dots, z_{-1}, z_0, z_1, \dots)$. We prove:

If F is C^1 with $\|\nabla_x F(x, z)\|_2 < 1$ and $\int_{\mathbb{R}^N \times \mathbb{R}^d} \max(1, \|\xi\|^4)|\widehat{F}_j(\xi)| d\xi < \infty$, then the RQNN system (1) induces a unique filter U^n with approximation error

$$\sup_z \sup_{t \in \mathbb{Z}_-} \|U(z)_t - U^n(z)_t\| \le \frac{C}{\sqrt{n}}, \quad \text{for all } n \text{ (with } C \text{ not depending on } n\text{).}$$

RQNNs approximate a large class of dynamical systems with error $\mathcal{O}(n^{-1/2})$.

Main Result II: Universality

We prove: RQNNs are universal approximators for sequence-to-sequence maps beyond state-space systems (3).

Consider general sequence-to-sequence maps U that are

- causal: filter output $U(z)_t$ only depends on inputs z_k for $k \le t$,
- time-invariant: time-shifted inputs translate to time-shifted outputs,
- fading memory: continuous dependence on input sequence.

Universal approximation theorem

Let U be a causal, time-invariant, fading memory filter. Then, for any $\varepsilon > 0$ there exist $n, N \in \mathbb{N}$, preprocessing matrices $P_1, \dots, P_N \in \mathbb{R}^{N \times N}$, a readout $W \in \mathbb{R}^{m \times N}$, and circuit parameters θ such that the RQNN (1)–(2) with added preprocessing induces a unique filter U^n with

$$\sup_{z=(\dots,z_{-1},z_0)} \sup_{t \in \mathbb{Z}_-} \|U(z)_t - U^n(z)_t\| \le \varepsilon.$$

Importantly, this universality holds with linear readout layer (2). Preprocessing is added via $\bar{F}_{\theta,j}(x, z) := R - 2R[\mathbb{P}_{1,\theta^j}(P_j x, z) + \mathbb{P}_{2,\theta^j}(P_j x, z)]$.

Conclusions

This work provides the first quantitative universality guarantees for recurrent quantum neural networks (RQNNs).

1. RQNNs are universal approximators for general sequence-to-sequence maps, including dynamical systems.
2. RQNNs can approximate a large class of dynamical systems with favourable scaling, achieving error ε with $\mathcal{O}(\varepsilon^{-2})$ weights and $\mathcal{O}(\log(\varepsilon^{-1}))$ qubits.

In particular, the quantum circuits require only logarithmically many qubits.

Comparison to classical RNN approximation bounds: RQNNs achieve the same approximation rate for state-space system approximations [5], but the Fourier integrability condition required in the quantum case is *strictly weaker*.

Physical circuit implementation: In the feedforward case, our QNN architecture has been implemented with Rydberg atoms in [6]. Recent work (e.g. [7]) shows the structure can be efficiently implemented.

References

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